Derivations and automorphism groups of the original deformative Schrödinger-Virasoro algebras [1]

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Abstract In this paper, we determine the derivation algebra and the automorphism group of the original deformative Schrödinger-Virasoro algebras which is the semi-direct product Lie algebra of the Witt algebra and its tensor density module $\mathbf{I^g}(a,b)$.

Keywords derivation algebra; automorphism group; Lie algebra $\mathbf{W^g}(a,b)$; Lie algebra $\mathbf{W}(a,b)$; Witt algebra.

Subject Classification 17B65, 17B68

1. Introduction

The Witt algebra $\mathbf{W}$ is an infinite dimensional Lie algebra over the complex field $\mathbb{C}$, with basis $\{L_m \mid m\in\mathbb{Z}\}$ and the defining relations:

$$[L_m, L_n] = (m-n)L_{m+n}, \quad \text{for any } m,n\in\mathbb{Z}.$$

The unique nontrivial one-dimensional central extension of $\mathbf{W}$ is the Virasoro algebra **Vir**, which is closely related to Kac-Moody algebras and plays an important role in 2D conformal field theory. There exist different generalizations of the classical Witt algebra and the Virasoro algebra, which have been studied by many authors (see for example [4, 13, 20, 23, 21], etc.). In [17] and [13], a class of representations $\mathbf{I}(a,b) = \bigoplus\limits_{m\in\mathbb{Z}} \mathbb{C}I_m$ for $\mathbf{W}$ with two complex parameters a and b were introduced. The action of $\mathbf{W}$ on $\mathbf{I}(a,b)$ is given by

$$L_m \cdot I_n = -(n+a+bm)I_{m+n}, \quad \forall m,n\in\mathbb{Z}.$$

$\mathbf{I}(a,b)$ is the so called tensor density module. In [25], the representations of $W(2,2)$ have been studied in terms of vertex operators algebras. In [21], they consider a generation of the Witt algebra $\mathbf{W}$:

$$\mathbf{W}(a,b) = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}L_m \bigoplus_{n\in\mathbb{Z}} \mathbb{C}I_n$$

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and determine its structures. $\mathbf{W}(a,b)$ has been considered in the mathematical and physical literature in [18]. In this paper, we consider the following generalization of the Lie algebra $\mathbf{W}(a,b)$(reference to [21]).

Let $\mathbf{W}^{\mathbf{g}}(a,b)=\bigoplus\limits_{m\in\mathbb{Z}}\mathbb{C}L_m\bigoplus\limits_{n\in\mathbb{Z}}\mathbb{C}I_n\bigoplus\limits_{k\in\mathbb{Z}}\mathbb{C}Y_{k+1/2}$ equipped with the following brackets:

$$[L_m,L_n]=(m-n)L_{m+n}, \tag{1.1}$$

$$[L_m,I_n]=-(n+a+bm)I_{m+n}, \tag{1.2}$$

$$[L_m,Y_{n+1/2}]=-(n+\frac{1-m+a+bm}{2})Y_{m+n+1/2}, \tag{1.3}$$

$$[I_m,I_n]=0, \tag{1.4}$$

$$[Y_{m+1/2},Y_{n+1/2}]=(m-n)I_{m+n+1}, \tag{1.5}$$

$$[I_m,Y_{n+1/2}]=0. \tag{1.6}$$

for all $m,n\in\mathbb{Z}$. Then $\mathbf{W}^{\mathbf{g}}(a,b)$ is an infinite dimensional Lie algebra over the complex field $\mathbb{C}$. The Lie algebra $\mathbf{W}^{\mathbf{g}}(a,b)$ is actually the same as $\mathcal{L}_{a,b}$ which is called the original deformative Schrödinger-Virasoro algebras by [16] up to isomorphism. In [16], the second cohomology group of original deformative Schrödinger-Virasoro algebras were determined. It is clear that $\mathbf{W}^{\mathbf{g}}(a,b)\simeq\mathbf{W}\ltimes\mathbf{I}^{\mathbf{g}}(a,b)$, where $\mathbf{W}$ is the Witt algebra and $\mathbf{I}^{\mathbf{g}}(a,b)=\bigoplus\limits_{m\in\mathbb{Z}}\mathbb{C}I_m\bigoplus\limits_{n\in\mathbb{Z}}\mathbb{C}Y_{n+1/2}$ is an ideal of $\mathbf{W}^{\mathbf{g}}(a,b)$. In this paper, our aim is to determine the derivation algebra and the automorphism group of $\mathbf{W}^{\mathbf{g}}(a,b)$.

2. The derivation algebra of $\mathbf{W}^{\mathbf{g}}(a,b)$

Let G be a commutative group, $\mathfrak{g}=\bigoplus\limits_{g\in G}\mathfrak{g}_g$ a G-graded Lie algebra. A $\mathfrak{g}$-module V is called G-graded, if

$$V=\bigoplus_{g\in G}V_{g\in G},\quad \mathfrak{g}_gV_h\subseteq V_{g+h},\quad \forall\, g,h\in G.$$

Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module. A linear map $D:\mathfrak{g}\longrightarrow V$ is called a derivation, if for any $x,y\in\mathfrak{g}$,

$$D[x,y]=x.D(y)-y.D(x).$$

If there exists some $v\in V$ such that $D:x\mapsto x.v$, then D is called an inner derivation. Denote by $\mathrm{Der}(\mathfrak{g},V)$ the vector space of all derivations, $\mathrm{Inn}(\mathfrak{g},V)$ the vector space of all inner derivations. Set

$$\mathrm{H}^1(\mathfrak{g},V)=\mathrm{Der}(\mathfrak{g},V)/\mathrm{Inn}(\mathfrak{g},V).$$

Denote by $\mathrm{Der}(\mathfrak{g})$ the derivation algebra of $\mathfrak{g}$, $\mathrm{Inn}(\mathfrak{g})$ the vector space of all inner derivations of $\mathfrak{g}$.

In this section, we shall determine all the derivations of the Lie algebra $\mathbf{W^g}(a,b)$.

Lemma 2.1. *For the Lie algebra $\mathbf{W^g}(a,b)$ defined in (1.1)-(1.6), we have*
(1) *As Lie algebras, $\mathbf{W^g}(a,b) \simeq \mathbf{W^g}(a+k,b)$ for any $k \in\ 2Z$.*
(2) *$\mathbf{W^g}(a,b)$ is perfect.*
(3) *the center of $\mathbf{W^g}(a,b)$ is*

$$\mathrm{Cent}(\mathbf{W^g}(a,b)) = \begin{cases} \mathbb{C}I_0, & \text{if } (a,b)=(0,0), \\ 0, & \text{otherwises.} \end{cases}$$

(4) *$\mathbf{W^g}(a,b)$ has a natural $\frac{1}{2}\mathbb{Z}$-grading defined by*

$$\mathbf{W^g}(a,b) = \bigoplus_{m\in\mathbb{Z}} \mathbf{W^g}(a,b)_{\frac{m}{2}} = (\bigoplus_{m\in\mathbb{Z}} \mathbf{W^g}(a,b)_m) \bigoplus (\bigoplus_{m\in\mathbb{Z}} \mathbf{W^g}(a,b)_{m+\frac{1}{2}}),$$

where $\mathbf{W^g}(a,b)_m = \mathbb{C}L_m \bigoplus \mathbb{C}I_m, \mathbf{W^g}(a,b)_{m+\frac{1}{2}} = \mathbb{C}Y_{m+\frac{1}{2}}$.

Proof. It is straightforward to prove the lemma by the definition of $\mathbf{W^g}(a,b)$. □

Remark 2.2. *By* (1) *of Lemma 2.1, we may assume that $a=0$ or $a=1$, if $a\in\mathbb{Z}$.*

By Proposition 1.1 in [5], we have the following lemma.

Lemma 2.3.

$$\mathrm{Der}(\mathbf{W^g}(a,b)) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Der}(\mathbf{W^g}(a,b))_{\frac{n}{2}},$$

where $\mathrm{Der}(\mathbf{W^g}(a,b))_{\frac{n}{2}} \mathbf{W^g}(a,b)_{\frac{m}{2}} \subseteq \mathbf{W^g}(a,b)_{\frac{m+n}{2}}$ for all $m,n\in\mathbb{Z}$.

□

Lemma 2.4. *For any $0\neq n\in\mathbb{Z}, a\neq 1$, we have $\mathrm{H}^1(\mathbf{W^g}(a,b)_0, \mathbf{W^g}(a,b)_{\frac{n}{2}}) = 0$.*

Proof. We have to prove

$$\mathrm{H}^1(\mathbf{W^g}(a,b)_0, \mathbf{W^g}(a,b)_m) = 0,\ \forall m\in\mathbb{Z}, m\neq 0,$$

and

$$\mathrm{H}^1(\mathbf{W^g}(a,b)_0, \mathbf{W^g}(a,b)_{m+\frac{1}{2}}) = 0,\ \forall m\in\mathbb{Z}.$$

(1)For any nonzero integer m, let $\varphi : \mathbf{W^g}(a,b)_0 \longmapsto \mathbf{W^g}(a,b)_m$ be a derivation, then we may assume

$$\varphi(L_0) = a_1 L_m + b_1 I_m, \qquad \varphi(I_0) = a_2 L_m + b_2 I_m,$$

where $a_i, b_i \in \mathbb{C}, i=1,2$. Because $\varphi[L_0, I_0] = [\varphi(L_0), I_0] + [L_0, \varphi(I_0)]$, we get

$$a_2(m-a) = 0, a_1(a+bm) + b_2 m = 0. \tag{2.1}$$

Since $a=0$ or $a=1$ or $a\notin\mathbb{Z}$, we have $b_2=-\frac{a_1(a+bm)}{m}$ and the following several cases.
Case1. $a=0$ or $a\notin\mathbb{Z}$. We have $a_2=0$ and

$$\varphi(L_0)=a_1L_m+b_1I_m,\qquad \varphi(I_0)=-\frac{a_1(a+bm)}{m}I_m.$$

Set $E_m=-\dfrac{a_1}{m}L_m-\dfrac{b_1}{m+a}I_m$, we have $\varphi(L_0)=[L_0,E_m]$ and $\varphi(I_0)=[I_0,E_m]$, which suggests that $\varphi\in Inn(\mathbf{W^g}(a,b)_0,\mathbf{W^g}(a,b)_m)$.
Case2. $a=1$.
Subcase2.1: $m\neq 1$. By (2.1), we have $a_2=0$.
If $m\neq -1$. Set $E_m=-\dfrac{a_1}{m}L_m-\dfrac{b_1}{m+1}I_m$, we also have

$$\varphi(L_0)=[L_0,E_m]\qquad \varphi(I_0)=[I_0,E_m].$$

So $\varphi\in Inn(\mathbf{W^g}(a,b)_0,\mathbf{W^g}(a,b)_m)$.
If $m=-1$. Set $E_{-1}=-a_1L_{-1}$, we have

$$\varphi(L_0)=adE_{-1}(L_0)+b_1I_{-1}\qquad \varphi(I_0)=adE_{-1}(I_0).$$

Subcase2.2: $m=1$. By (2.1), we have $a_2\in\mathbb{C}$ and

$$\varphi(L_0)=a_1L_1+b_1I_1,\qquad \varphi(I_0)=a_2L_1-a_1(1+b)I_1.$$

Set $E_1=-a_1L_1-\dfrac{b_1}{2}I_1$, we have

$$\varphi(L_0)=ad(-E_1)(L_0)\qquad \varphi(I_0)=ad(-E_1)(I_0)+a_2L_1.$$

(2) For all $m\in\mathbb{Z}$, let $\varphi:\mathbf{W^g}(a,b)_0\longmapsto\mathbf{W^g}(a,b)_{m+\frac{1}{2}}$ be a derivation, then we may assume

$$\varphi(L_0)=a_1Y_{m+\frac{1}{2}},\qquad \varphi(I_0)=b_1Y_{m+\frac{1}{2}},$$

where $a_1,b_1\in\mathbb{C}$. Since $\varphi[L_0,I_0]=[\varphi(L_0),I_0]+[L_0,\varphi(I_0)]$, we have

$$\varphi(-aI_0)=-ab_1Y_{m+\frac{1}{2}}=[L_0,b_1Y_{m+\frac{1}{2}}]=-b_1(m+\frac{1+a}{2})Y_{m+\frac{1}{2}},$$

then

$$b_1[\frac{a}{2}-(m+\frac{1}{2})]Y_{m+\frac{1}{2}}=0. \tag{2.2}$$

Case1. $a=0$ or $a\notin\mathbb{Z}$. By (2.2), we have $b_1=0$ and

$$\varphi(L_0)=a_1Y_{m+\frac{1}{2}},\qquad \varphi(I_0)=0$$

Set $E_{m+\frac{1}{2}}=\dfrac{a_1}{m+\dfrac{1+a}{2}}Y_{m+\frac{1}{2}}$, we have

$$\varphi(L_0)=a_1Y_{m+\frac{1}{2}}=[E_{m+\frac{1}{2}},L_0],\quad \varphi(I_0)=0=[E_{m+\frac{1}{2}},I_0].$$

So $\varphi \in Inn(\mathbf{W}^{\mathbf{g}}(a,b)_0, \mathbf{W}^{\mathbf{g}}(a,b)_{m+\frac{1}{2}})$.
Case2. $a = 1$. By (2.2), if $m \neq 0$, we have $b_1 = 0$ and

$$\varphi(L_0) = a_1 Y_{m+\frac{1}{2}}, \qquad \varphi(I_0) = 0$$

If $m \neq -1$, set $E_{m+\frac{1}{2}} = \dfrac{a_1}{m+1} Y_{m+\frac{1}{2}}$, we have

$$\varphi(L_0) = [E_{m+\frac{1}{2}}, L_0] = adE_{m+\frac{1}{2}}(L_0), \quad \varphi(I_0) = 0 = adE_{m+\frac{1}{2}}(I_0).$$

If $m = -1$. By the relations of brackets for $\mathbf{W}^{\mathbf{g}}(a,b)$, we know φ is an outer derivation.
By (2.2), if $m = 0$, we have $b_1 \in \mathbb{C}$ and

$$\varphi(L_0) = a_1 Y_{\frac{1}{2}}, \qquad \varphi(I_0) = b_1 Y_{\frac{1}{2}}.$$

Set $E_{\frac{1}{2}} = a_1 Y_{\frac{1}{2}}$, we have

$$\varphi(L_0) = [E_{\frac{1}{2}}, L_0] = adE_{\frac{1}{2}}(L_0), \quad \varphi(I_0) = adE_{\frac{1}{2}}(I_0) + b_1 Y_{\frac{1}{2}}.$$

This completes the proof of lemma. □

Lemma 2.5. $\mathrm{Hom}_{\mathbf{W}^{\mathbf{g}}(a,b)_0}(\mathbf{W}^{\mathbf{g}}(a,b)_{\frac{m}{2}}, \mathbf{W}^{\mathbf{g}}(a,b)_{\frac{n}{2}}) = 0$ *for all* $m, n \in \mathbb{Z}$, $m \neq n, a \neq 1$.

Proof. We have to consider the following four identities:

$$\mathrm{Hom}_{\mathbf{W}^{\mathbf{g}}(a,b)_0}(\mathbf{W}^{\mathbf{g}}(a,b)_m, \mathbf{W}^{\mathbf{g}}(a,b)_n) = 0, \quad m \neq n, m, n \in \mathbb{Z}; \tag{2.3}$$

$$\mathrm{Hom}_{\mathbf{W}^{\mathbf{g}}(a,b)_0}(\mathbf{W}^{\mathbf{g}}(a,b)_m, \mathbf{W}^{\mathbf{g}}(a,b)_{n+\frac{1}{2}}) = 0, \qquad m, n \in \mathbb{Z}; \tag{2.4}$$

$$\mathrm{Hom}_{\mathbf{W}^{\mathbf{g}}(a,b)_0}(\mathbf{W}^{\mathbf{g}}(a,b)_{m+\frac{1}{2}}, \mathbf{W}^{\mathbf{g}}(a,b)_n) = 0, \qquad m, n \in \mathbb{Z}; \tag{2.5}$$

$$\mathrm{Hom}_{\mathbf{W}^{\mathbf{g}}(a,b)_0}(\mathbf{W}^{\mathbf{g}}(a,b)_{m+\frac{1}{2}}, \mathbf{W}^{\mathbf{g}}(a,b)_{n+\frac{1}{2}}) = 0, m \neq n, m, n \in \mathbb{Z}. \tag{2.6}$$

For integers $m \neq n$, let $f \in \mathrm{Hom}_{\mathbf{W}^{\mathbf{g}}(a,b)_0}(\mathbf{W}^{\mathbf{g}}(a,b)_{\frac{m}{2}}, \mathbf{W}^{\mathbf{g}}(a,b)_{\frac{n}{2}})$, then for $E_0 \in \mathbf{W}^{\mathbf{g}}(a,b)_0$, $E_m \in \mathbf{W}^{\mathbf{g}}(a,b)_{\frac{m}{2}}$, we have

$$f([E_0, E_{\frac{m}{2}}]) = [E_0, f(E_{\frac{m}{2}})]. \tag{2.7}$$

(1)It follows from $f([L_0, L_m]) = [L_0, f(L_m)]$ that $f(-mL_m) = [L_0, f(L_m)]$. Assume $f(L_m) = a_1 L_n + b_1 I_n$, we have $-ma_1 L_n - mb_1 I_n = -a_1 n L_n - b_1 n I_n - b_1 a I_n$. Since $m \neq n$, we get

$$a_1 = 0, \qquad (m - n - a) b_1 = 0. \tag{2.8}$$

For $a = 0$ or $a \notin \mathbb{Z}$, we have $b_1 = 0$ and $f(L_m) = 0$. On the other hand, we have $f([L_0, I_m]) = [L_0, f(I_m)]$. Similarly, we have $f(I_m) = 0$. Therefore, $\mathrm{Hom}_{\mathbf{W}^{\mathbf{g}}(a,b)_0}(\mathbf{W}^{\mathbf{g}}(a,b)_m, \mathbf{W}^{\mathbf{g}}(a,b)_n) = 0$.
For $a = 1$. By (2.8), we get

$$b_1 = 0, \quad m - n \neq 1; \quad b_1 \in \mathbb{C}, \quad m - n = 1.$$

So $f(L_m) = b_1 I_{m-1}$. Consequently (2.3) holds.
(2)By $f([L_0, L_m]) = [L_0, f(L_m)]$ and $f([L_0, I_m]) = [L_0, f(I_m)]$ we have

$$mf(L_m) = (n + \frac{1+a}{2})f(L_m), \tag{2.9}$$

$$(m+a)f(I_m) = (n + \frac{1+a}{2})f(I_m). \tag{2.10}$$

For $a = 0$ or $a \notin \mathbb{Z}$, obviously, $f(L_m) = 0$ and $f(I_m) = 0$.
For $a = 1$, by (2.9) and (2.10), we have

$$f(L_m) = 0, \quad m - n \neq 1; \quad f(I_m) = 0, \quad m \neq n.$$

Then

$$\mathrm{Hom}_{\mathbf{W}^{\mathbf{g}}(a,b)_0}(\mathbf{W}^{\mathbf{g}}(a,b)_m, \mathbf{W}^{\mathbf{g}}(a,b)_{n+\frac{1}{2}}) = 0, \quad a \neq 1,$$

which shows (2.4).
(3) By (2.7), we have $f([L_0, Y_{m+\frac{1}{2}}]) = [L_0, f(Y_{m+\frac{1}{2}})]$. Assume $f(Y_{m+\frac{1}{2}}) = x_n L_n + y_n I_n$, we have

$$-(m + \frac{1+a}{2})(x_n L_n + y_n I_n) = -x_n n L_n - y_n(n+a)I_n. \tag{2.11}$$

For $a = 0$ or $a \notin \mathbb{Z}$, we have $x_n = 0, y_n = 0$. Then $f(Y_{m+\frac{1}{2}}) = 0$. For $a = 1$, by (2.11), we have

$$(m+1-n)x_n = 0, \quad (m-n)y_n = 0.$$

So

$$x_n = 0, \quad m - n \neq -1; \quad y_n = 0, \quad m \neq n$$

Therefore

$$\mathrm{Hom}_{\mathbf{W}^{\mathbf{g}}(a,b)_0}(\mathbf{W}^{\mathbf{g}}(a,b)_{m+\frac{1}{2}}, \mathbf{W}^{\mathbf{g}}(a,b)_n) = 0, a \neq 1.$$

This shows (2.5).
(4) Assume $f(Y_{m+\frac{1}{2}}) = x_n Y_{n+\frac{1}{2}}$, by $f([L_0, Y_{m+\frac{1}{2}}]) = [L_0, f(Y_{m+\frac{1}{2}})]$, we have

$$(m + \frac{1+a}{2})f(Y_{m+\frac{1}{2}}) = (n + \frac{1+a}{2})f(Y_{m+\frac{1}{2}}).$$

Since $m \neq n$, we get $f(Y_{m+\frac{1}{2}}) = 0$. So

$$\mathrm{Hom}_{\mathbf{W}^{\mathbf{g}}(a,b)_0}(\mathbf{W}^{\mathbf{g}}(a,b)_{m+\frac{1}{2}}, \mathbf{W}^{\mathbf{g}}(a,b)_{n+\frac{1}{2}}) = 0, m \neq n.$$

This proves (2.6). □

By Lemma 2.4-2.5 and Proposition 1.2 in [5], we have the following Lemma.

Lemma 2.6.

$$\begin{aligned}\mathrm{Der}(\mathbf{W}^{\mathbf{g}}(a,b), \mathbf{W}^{\mathbf{g}}(a,b)) \;&=\; \mathrm{Der}(\mathbf{W}^{\mathbf{g}}(a,b))_0 + Inn(\mathbf{W}^{\mathbf{g}}(a,b)) + \mathrm{Der}(\mathbf{W}^{\mathbf{g}}(a,b))_1 \\ &\quad + \mathrm{Der}(\mathbf{W}^{\mathbf{g}}(a,b))_{-1} + \mathrm{Der}(\mathbf{W}^{\mathbf{g}}(a,b))_{\frac{1}{2}} + \mathrm{Der}(\mathbf{W}^{\mathbf{g}}(a,b))_{-\frac{1}{2}}.\end{aligned}$$

Lemma 2.7. (1) *Up to isomorphism, for* $(a,b)\notin\{(0,0),(0,1),(0,2)\}$, *we have*

$$\mathrm{H}^1(\mathbf{W^g}(a,b)_{\frac{m}{2}},\mathbf{W^g}(a,b)_{\frac{m}{2}})=\mathbb{C}\bar{D},$$

where

$$\bar{D}(L_m)=0,\quad \bar{D}(I_m)=I_m,\quad \bar{D}(Y_{m+\frac{1}{2}})=Y_{m+\frac{1}{2}},\quad \forall\, m\in\mathbb{Z}.$$

(2) $\mathrm{H}^1(\mathbf{W^g}(0,0)_{\frac{m}{2}},\mathbf{W^g}(0,0)_{\frac{m}{2}})=\mathbb{C}\bar{D}_1\bigoplus\mathbb{C}\bar{D}_2\bigoplus\mathbb{C}\bar{D}_3$, *where*

$$\bar{D}_1(L_m)=mI_m,\quad \bar{D}_1(I_m)=0,\quad \bar{D}_1(Y_{m+\frac{1}{2}})=0,$$
$$\bar{D}_2(L_m)=(m-1)I_m,\quad \bar{D}_2(I_m)=0,\quad \bar{D}_2(Y_{m+\frac{1}{2}})=0,$$
$$\bar{D}_3(L_m)=0,\quad \bar{D}_3(I_m)=I_m,\quad \bar{D}_3(Y_{m+\frac{1}{2}})=Y_{m+\frac{1}{2}},\quad \forall\, m\in\mathbb{Z}.$$

(3) $\mathrm{H}^1(\mathbf{W^g}(0,1)_{\frac{m}{2}},\mathbf{W^g}(0,1)_{\frac{m}{2}})=\mathbb{C}\bar{D}_1\bigoplus\mathbb{C}\bar{D}_2$, *where*

$$\bar{D}_1(L_m)=0,\quad \bar{D}_1(I_m)=I_m,\quad \bar{D}_1(Y_{m+\frac{1}{2}})=Y_{m+\frac{1}{2}};$$
$$\bar{D}_2(L_m)=m(m-1)I_m,\quad \bar{D}_2(I_m)=0,\quad \bar{D}_2(Y_{m+\frac{1}{2}})=0\forall\, m\in\mathbb{Z}.$$

(4) $\mathrm{H}^1(\mathbf{W^g}(0,2)_{\frac{m}{2}},\mathbf{W^g}(0,2)_{\frac{m}{2}})=\mathbb{C}\bar{D}_1\bigoplus\mathbb{C}\bar{D}_2$, *where*

$$\bar{D}_1(L_m)=0,\quad \bar{D}_1(I_m)=I_m,\quad \bar{D}_1(Y_{m+\frac{1}{2}})=Y_{m+\frac{1}{2}};$$
$$\bar{D}_2(L_m)=m^3I_m,\quad \bar{D}_2(I_m)=0,\quad \bar{D}_2(Y_{m+\frac{1}{2}})=0\ \forall\, m\in\mathbb{Z}.$$

Proof. For $D\in\mathrm{Der}(\mathbf{W^g}(a,b),\mathbf{W^g}(a,b))_0$, assume that

$$D(L_m)=a_{11}^mL_m+a_{12}^mI_m,\quad D(I_m)=a_{21}^mL_m+a_{22}^mI_m,\quad D(Y_{m+\frac{1}{2}})=b^{m+\frac{1}{2}}Y_{m+\frac{1}{2}},$$

for all $m\in\mathbb{Z}$, where $a_{ij}^m, b^{m+\frac{1}{2}}\in\mathbb{C}, i,j=1,2$. By the definition of derivation and the product in $\mathbf{W^g}(a,b)$, we get

$$(m-n)a_{11}^{m+n}=(m-n)a_{11}^m+(m-n)a_{11}^n, \tag{2.12}$$
$$(m-n)a_{12}^{m+n}=(m+a+bn)a_{12}^m-(n+a+bm)a_{12}^n, \tag{2.13}$$
$$-(n+a+bm)a_{21}^{m+n}=(m-n)a_{21}^n, \tag{2.14}$$
$$(n+a+bm)a_{22}^{m+n}=(n+a+bm)a_{11}^m+(n+a+bm)a_{22}^n, \tag{2.15}$$
$$(n+\frac{1-m+a+bm}{2})b^{m+n+\frac{1}{2}}=(n+\frac{1-m+a+bm}{2})(a_{11}^m+b^{n+\frac{1}{2}}), \tag{2.16}$$
$$(m+a+bn)a_{21}^n=(n+a+bm)a_{21}^m, \tag{2.17}$$
$$(n+\frac{1-m+a+bm}{2})a_{21}^m=0, \tag{2.18}$$
$$(m-n)a_{21}^{m+n+1}=0, \tag{2.19}$$
$$(m-n)a_{22}^{m+n+1}=(m-n)b^{m+\frac{1}{2}}+(m-n)b^{n+\frac{1}{2}}, \tag{2.20}$$

for all $m,n\in\mathbb{Z}$. Let $n=0$ in (2.12)-(2.15), we have

$$a_{11}^0=0, \tag{2.21}$$

$$(a+bm)a_{12}^0=aa_{12}^m, \tag{2.22}$$

$$-(a+bm)a_{21}^m=ma_{21}^0, \tag{2.23}$$

$$(a+bm)a_{22}^m=(a+bm)(a_{11}^m+a_{22}^0). \tag{2.24}$$

On the other hand, let $m=0$ in (2.14) and (2.18), we get

$$aa_{21}^n=0,\quad a_{21}^0(n+\frac{1+a}{2})=0 \tag{2.25}$$

It follows from (2.12), (2.19) and (2.20) that

$$a_{11}^{m+n}=a_{11}^m+a_{11}^n,\quad m\neq n, \tag{2.26}$$

$$a_{21}^{m+n+1}=0,\quad m\neq n,$$

$$a_{22}^{m+n+1}=b^{m+\frac{1}{2}}+b^{n+\frac{1}{2}},\quad m\neq n. \tag{2.27}$$

Let $m=-n\neq 0$ in (2.26) and combine (2.21), we have

$$a_{11}^{-m}=-a_{11}^m,\ \ \forall m\in\mathbb{Z}.$$

On the other hand, let $n=1$ in (2.26), we have

$$a_{11}^{m+1}=a_{11}^m+a_{11}^1,\quad m\neq 1. \tag{2.28}$$

By induction on $m\in\mathbb{Z}^+$ in (2.28), we get

$$a_{11}^m=a_{11}^2+(m-2)a_{11}^1,\quad m\neq 1. \tag{2.29}$$

Let $m=4,n=-2$ in (2.26), we have

$$a_{11}^2=2a_{11}^1. \tag{2.30}$$

Combine (2.29) and (2.30), we get

$$a_{11}^m=ma_{11}^1,\quad \forall m\in\mathbb{Z}.$$

Let $m=0$ in (2.19), we have $a_{21}^{n+1}=0,\ \ n\neq 0$. Moreover let $n=-m\neq 0$ in (2.19), We also have $a_{21}^1=0$. So we get

$$a_{21}^m=0,\quad \forall m\in\mathbb{Z}.$$

Case 1: $a\notin\mathbb{Z}$. By (2.22), we have $a_{12}^m=\dfrac{a+bm}{a}a_{12}^0$ for all $m\in\mathbb{Z}$.
Subcase 1.1: If $bm+a\neq 0$ for all $m\in\mathbb{Z}$, By (2.24), we have

$$a_{22}^m=a_{11}^m+a_{22}^0=ma_{11}^1+a_{22}^0,\ \ \forall m\in\mathbb{Z}$$

. Let $n=0$ in (2.27), we have

$$b^{m+\frac{1}{2}}=a_{22}^{m+1}-b^{\frac{1}{2}}=(m+1)a_{11}^{1}+a_{22}^{0}-b^{\frac{1}{2}},\ m\neq 0. \tag{2.31}$$

On the other hand, let $n=0, m=1$ in (2.16) and use (2.31), we have

$$b^{\frac{1}{2}}=b^{1+\frac{1}{2}}-a_{11}^{1}=a_{11}^{1}+a_{22}^{0}-b^{\frac{1}{2}} \tag{2.32}$$

So

$$b^{\frac{1}{2}}=\frac{1}{2}(a_{11}^{1}+a_{22}^{0}).$$

Combine (2.31) and (2.32), we get

$$b^{m+\frac{1}{2}}=ma_{11}^{1}+b^{\frac{1}{2}}=(m+\frac{1}{2})a_{11}^{1}+\frac{1}{2}a_{22}^{0},\quad \forall m\in\mathbb{Z}.$$

Then

$$D(L_m)=ma_{11}^{1}L_m+\frac{a+bm}{a}a_{12}^{0}I_m,\quad D(I_m)=(ma_{11}^{1}+a_{22}^{0})I_m,$$

$$D(Y_{m+\frac{1}{2}})=[(m+\frac{1}{2})a_{11}^{1}+\frac{1}{2}a_{22}^{0}]Y_{m+\frac{1}{2}},$$

for all $m\in\mathbb{Z}$. Set $E_0=-a_{11}^{1}L_0+\dfrac{a_{12}^{0}}{a}I_0$, then

$$D(L_m)=adE_0(L_m),\quad D(I_m)=adE_0(I_m)+(a_{22}^{0}-aa_{11}^{1})I_m,$$

$$D(Y_{m+\frac{1}{2}})=adE_0(Y_{m+\frac{1}{2}})+\frac{1}{2}(a_{22}^{0}-aa_{11}^{1})Y_{m+\frac{1}{2}},$$

for all $m\in\mathbb{Z}$. Let $\bar{D}\in \mathrm{H}^1(\mathbf{W}^{\mathbf{g}}(a,b)_{\frac{m}{2}},\mathbf{W}^{\mathbf{g}}(a,b)_{\frac{m}{2}})$ such that

$$\bar{D}(L_m)=0,\bar{D}(I_m)=I_m,\bar{D}(Y_{m+\frac{1}{2}})=Y_{m+\frac{1}{2}}$$

for all $m\in\mathbb{Z}$, then we have

$$\mathrm{H}^1(\mathbf{W}^{\mathbf{g}}(a,b)_{\frac{m}{2}},\mathbf{W}^{\mathbf{g}}(a,b)_{\frac{m}{2}})=\mathbb{C}\bar{D}.$$

Subcase 1.2: If there exists some $\mu\in\mathbb{Z}$ such that $b\mu+a=0$, then it follows from (2.24) that

$$a_{22}^{m}=a_{11}^{m}+a_{22}^{0}=ma_{11}^{1}+a_{22}^{0},\quad m\neq\mu.$$

Obviously, $\mu\neq 0$, (or else $a=0$, a contradiction.) Letting $m=n=\mu$ in (2.15), we get

$$a_{22}^{\mu}=a_{22}^{2\mu}-a_{11}^{\mu}=2\mu a_{11}^{1}+a_{22}^{0}-a_{11}^{\mu}=\mu a_{11}^{1}+a_{22}^{0}.$$

Therefore, we still have $a_{22}^{m}=ma_{11}^{1}+a_{22}^{0}$ for all $m\in\mathbb{Z}$. Let $m=\mu, n=-\mu$ in (2.16) and use (2.31), we have

$$b^{\frac{1}{2}}=\frac{1}{2}(a_{11}^{1}+a_{22}^{0}).$$

According to (2.31), we still get $b^{m+\frac{1}{2}}=(m+\frac{1}{2})a_{11}^{1}+\frac{1}{2}a_{22}^{0}$ for all $m\in\mathbb{Z}$. Then we have the same result as Subcase 1.1.

Case 2: $a = 0$. From (2.22) we get $ba_{12}^0 = 0$. Obviously, $a_{12}^0 = 0$ when $b \neq 0$. Let $n = -m$ in (2.13), then we have

$$(1-b)[a_{12}^m + a_{12}^{-m}] = 2a_{12}^0, \qquad m \neq 0. \tag{2.33}$$

Subcase 2.1: $b = 0$. From (2.13) and (2.15), we have

$$(m-n)a_{12}^{m+n} = ma_{12}^m - na_{12}^n, \tag{2.34}$$

$$a_{22}^{m+n} = a_{11}^m + a_{22}^n, \qquad n \neq 0. \tag{2.35}$$

Let $n = 1$ in (2.34) and (2.35) respectively, we have

$$(m-1)a_{12}^{m+1} = ma_{12}^m - a_{12}^1. \tag{2.36}$$

$$a_{22}^{m+1} = a_{11}^m + a_{22}^1 = ma_{11}^1 + a_{22}^1, \quad \forall m \in \mathbb{Z}. \tag{2.37}$$

On the other hand, let $n = 1, m = -1$ in (2.35), we have

$$a_{22}^1 = a_{11}^1 + a_{22}^0. \tag{2.38}$$

From (2.37) and (2.38), it is easy to deduce that

$$a_{22}^m = ma_{11}^1 + a_{22}^0, \quad \forall m \in \mathbb{Z}.$$

It follows from (2.33) that

$$a_{12}^{-m} = 2a_{12}^0 - a_{12}^m, \qquad \forall\, m \in \mathbb{Z}. \tag{2.39}$$

By induction on $m > 1$ in (2.36), we can deduce that

$$a_{12}^m = (m-1)a_{12}^2 - (m-2)a_{12}^1, \qquad m > 0.$$

Set $m = -2$ in (2.36) and use (2.39), then we get

$$a_{12}^0 = 2a_{12}^1 - a_{12}^2.$$

Combine the two identities, we have

$$a_{12}^m = ma_{12}^1 - (m-1)a_{12}^0, \quad \forall m \in \mathbb{Z}.$$

Let $n = 0, m = 2$ in (2.16) and use (2.31), we have

$$b^{\frac{1}{2}} = \frac{1}{2}(a_{11}^1 + a_{22}^0).$$

So we get

$$b^{m+\frac{1}{2}} = ma_{11}^1 + b^{\frac{1}{2}} = (m + \frac{1}{2})a_{11}^1 + \frac{1}{2}a_{22}^0, \quad \forall m \in \mathbb{Z}.$$

Therefore,

$$D(L_m) = ma_{11}^1 L_m + [ma_{12}^1 - (m-1)a_{12}^0]I_m,$$

$$D(I_m) = (a_{22}^0 + ma_{11}^1)I_m, \quad D(Y_{m+\frac{1}{2}}) = [(m + \frac{1}{2})a_{11}^1 + \frac{1}{2}a_{22}^0]Y_{m+\frac{1}{2}},$$

for all $m \in \mathbb{Z}$. Setting $E_0 = -a_{11}^1 L_0$, we can deduce that

$$D(L_m) = adE_0(L_m) + [ma_{12}^1 - (m-1)a_{12}^0]I_m,$$

$$D(I_m) = adE_0(I_m) + a_{22}^0 I_m, \quad D(Y_{m+\frac{1}{2}}) = adE_0(Y_{m+\frac{1}{2}}) + \frac{1}{2}a_{22}^0 Y_{m+\frac{1}{2}}.$$

Let $\bar{D}_1, \bar{D}_2, \bar{D}_3 \in \mathrm{H}^1(\mathbf{W}^{\mathbf{g}}(0,0)_{\frac{m}{2}}, \mathbf{W}^{\mathbf{g}}(0,0)_{\frac{m}{2}})$ such that

$$\bar{D}_1(L_m) = mI_m, \qquad \bar{D}_1(I_m) = 0, \qquad \bar{D}_1(Y_{m+\frac{1}{2}}) = 0;$$

$$\bar{D}_2(L_m) = (m-1)I_m, \qquad \bar{D}_2(I_m) = 0, \qquad \bar{D}_2(Y_{m+\frac{1}{2}}) = 0;$$

$$\bar{D}_3(L_m) = 0, \qquad \bar{D}_3(I_m) = I_m, \qquad \bar{D}_3(Y_{m+\frac{1}{2}}) = Y_{m+\frac{1}{2}},$$

for all $m \in \mathbb{Z}$, then they are all outer derivations and

$$\mathrm{H}^1(\mathbf{W}^{\mathbf{g}}(0,0)_{\frac{m}{2}}, \mathbf{W}^{\mathbf{g}}(0,0)_{\frac{m}{2}}) = \mathbb{C}\bar{D}_1 \bigoplus \mathbb{C}\bar{D}_2 \bigoplus \mathbb{C}\bar{D}_3.$$

Subcase 2.2: $b = 1$. Then $a_{12}^0 = 0$. By (2.13) and (2.15), we have

$$(m-n)a_{12}^{m+n} = (m+n)[a_{12}^m - a_{12}^n], \qquad m, n \in \mathbb{Z}. \tag{2.40}$$

$$a_{22}^{m+n} = a_{11}^m + a_{22}^n, \qquad m+n \neq 0. \tag{2.41}$$

Letting $n = \pm 1$ respectively in (2.40) and using induction on m, we can deduce

$$a_{12}^m = \frac{m(m-1)}{2}a_{12}^{-1} + \frac{m(m+1)}{2}a_{12}^1 = \frac{m(m-1)}{2}[a_{12}^{-1} + a_{12}^1] + ma_{12}^1,$$

for all $m \in \mathbb{Z}$. It is easy following (2.41) to see that

$$a_{22}^m = a_{11}^m + a_{22}^0 = ma_{11}^1 + a_{22}^0, \quad m \neq 0.$$

So we get

$$a_{22}^m = ma_{11}^1 + a_{22}^0, \quad \forall m \in \mathbb{Z}.$$

Letting $n = 0$ in (2.16), we get

$$b^{m+\frac{1}{2}} = ma_{11}^1 + b^{\frac{1}{2}} = (m + \frac{1}{2})a_{11}^1 + \frac{1}{2}a_{22}^0,$$

for all $m \in \mathbb{Z}$. Therefore,

$$D(L_m) = ma_{11}^1 L_m + [\frac{m(m-1)}{2}a_{12}^{-1} + \frac{m(m+1)}{2}a_{12}^1]I_m,$$

$$D(I_m) = (ma_{11}^1 + a_{22}^0)I_m, \quad D(Y_{m+\frac{1}{2}}) = [(m+\frac{1}{2})a_{11}^1 + \frac{1}{2}a_{22}^0]Y_{m+\frac{1}{2}}.$$

Set $E_0 = -a_{11}^1 L_0 + a_{12}^1 I_0$, then

$$D(L_m) = adE_0(L_m) + \frac{m(m-1)}{2}[a_{12}^{-1} + a_{12}^1]I_m,$$

$$D(I_m) = adE_0(I_m) + a_{22}^0 I_m, \quad D(Y_{m+\frac{1}{2}}) = adE_0(Y_{m+\frac{1}{2}}) + \frac{1}{2}a_{22}^0 Y_{m+\frac{1}{2}}.$$

Let $\bar{D}_1, \bar{D}_2 \in \mathrm{H}^1(\mathbf{W}^{\mathbf{g}}(0,1)_{\frac{m}{2}}, \mathbf{W}^{\mathbf{g}}(0,1)_{\frac{m}{2}})$ such that

$$\bar{D}_1(L_m) = 0, \qquad \bar{D}_1(I_m) = I_m, \quad \bar{D}_1(Y_{m+\frac{1}{2}}) = Y_{m+\frac{1}{2}};$$
$$\bar{D}_2(L_m) = m(m-1)I_m, \quad \bar{D}_2(I_m) = 0, \quad \bar{D}_2(Y_{m+\frac{1}{2}}) = 0,$$

for all $m \in \mathbb{Z}$, then we have

$$\mathrm{H}^1(\mathbf{W}^{\mathbf{g}}(0,1)_{\frac{m}{2}}, \mathbf{W}^{\mathbf{g}}(0,1)_{\frac{m}{2}}) = \mathbb{C}\bar{D}_1 \bigoplus \mathbb{C}\bar{D}_2.$$

Subcase 2.3: $b \neq 0, 1$. Then we always have $a_{12}^{-m} = -a_{12}^m$ from (2.33) for all $m \in \mathbb{Z}$. Let $n = 1$ in (2.13), we have

$$(m-1)a_{12}^{m+1} = (b+m)a_{12}^m - (bm+1)a_{12}^1.$$

Then

$$(m-1)[a_{12}^{m+1} - (m+1)a_{12}^1] = (m+b)[a_{12}^m - ma_{12}^1]. \tag{2.42}$$

Let $m = -2$ in (2.42), we have

$$(b-2)[a_{12}^2 - 2a_{12}^1] = 0.$$

Hence

$$a_{12}^2 = 2a_{12}^1, \quad b \neq 2.$$

By induction on m in (2.42), we can deduce that

$$a_{12}^m = ma_{12}^1, \quad \forall m \in \mathbb{Z}.$$

By (2.24), we have

$$a_{22}^m = a_{11}^m + a_{22}^0 = ma_{11}^1 + a_{22}^0, \quad m \in \mathbb{Z}.$$

Similar to the computations in subcase1.1, we have

$$b^{m+\frac{1}{2}} = ma_{11}^1 + b^{\frac{1}{2}} = (m + \frac{1}{2})a_{11}^1 + \frac{1}{2}a_{22}^0, \quad \forall m \in \mathbb{Z}.$$

So

$$D(L_m) = ma_{11}^1 L_m + ma_{12}^1 I_m, \quad D(I_m) = (ma_{11}^1 + a_{22}^0)I_m,$$
$$D(Y_{m+\frac{1}{2}}) = [(m + \frac{1}{2})a_{11}^1 + \frac{1}{2}a_{22}^0]Y_{m+\frac{1}{2}}.$$

Set $E_0 = -a_{11}^1 L_0 + \dfrac{a_{12}^1}{b} I_0$, then

$$D(L_m) = adE_0(L_m), \quad D(I_m) = adE_0(I_m) + a_{22}^0 I_m,$$
$$D(Y_{m+\frac{1}{2}}) = adE_0(Y_{m+\frac{1}{2}}) + \frac{1}{2}a_{22}^0 Y_{m+\frac{1}{2}}.$$

Consequently, for $b \neq 0, 1, 2$, we have

$$\mathrm{H}^1(\mathbf{W}^{\mathbf{g}}(0,b)_{\frac{m}{2}}, \mathbf{W}^{\mathbf{g}}(0,b)_{\frac{m}{2}}) = \mathbb{C}\bar{D},$$

where $\bar{D}(L_m) = 0, \quad \bar{D}(I_m) = I_m, \quad \bar{D}(Y_{m+\frac{1}{2}}) = Y_{m+\frac{1}{2}}$ for all $m \in \mathbb{Z}$.

If $b=2$, by induction on m in (2.42), we have

$$a_{12}^{m}=\frac{m^3-m}{6}a_{12}^{2}-\frac{m^3-4m}{3}a_{12}^{1}=\frac{a_{12}^{2}-2a_{12}^{1}}{6}m^3-\frac{a_{12}^{2}-8a_{12}^{1}}{6}m,$$

for all $m\in\mathbb{Z}$. Then

$$D(L_m)=ma_{11}^{1}L_m+(\frac{a_{12}^{2}-2a_{12}^{1}}{6}m^3-\frac{a_{12}^{2}-8a_{12}^{1}}{6}m)I_m,\quad D(I_m)=(ma_{11}^{1}+a_{22}^{0})I_m,$$

$$D(Y_{m+\frac{1}{2}})=[(m+\frac{1}{2})a_{11}^{1}+\frac{1}{2}a_{22}^{0}]Y_{m+\frac{1}{2}}.$$

Set $E_0=-a_{11}^{1}L_0-\dfrac{a_{12}^{2}-8a_{12}^{1}}{12}I_0$, then

$$D(L_m)=adE_0(L_m)+\frac{a_{12}^{2}-2a_{12}^{1}}{6}m^3I_m,\quad D(I_m)=adE_0(I_m)+a_{22}^{0}I_m,$$

$$D(Y_{m+\frac{1}{2}})=adE_0(Y_{m+\frac{1}{2}})+\frac{1}{2}a_{22}^{0}Y_{m+\frac{1}{2}}.$$

Consequently, we have

$$\mathrm{H}^1(\mathbf{W^g}(0,2)_{\frac{m}{2}},\mathbf{W^g}(0,2)_{\frac{m}{2}})=\mathbb{C}\bar{D}_1\bigoplus\mathbb{C}\bar{D}_2,$$

where $\bar{D}_1(L_m)=0,\ \bar{D}_1(I_m)=I_m,\ \bar{D}_1(Y_{m+\frac{1}{2}})=Y_{m+\frac{1}{2}};\ \bar{D}_2(L_m)=m^3I_m,\ \bar{D}_2(I_m)=0,$ $\bar{D}_2(Y_{m+\frac{1}{2}})=0$ for all $m\in\mathbb{Z}$.

Case 3: $a=1$. Similar to the above discussions for the case $a\notin\mathbb{Z}$ completely, we only need to take $a=1$ in these discussions and get the same results as case 1. $\square$

Lemma 2.8. *Up to isomorphism,*

$$\mathrm{H}^1(\mathbf{W^g}(a,b)_{\frac{m}{2}},\mathbf{W^g}(a,b)_{\frac{m}{2}\pm1})=0.$$

Proof. By [21], $\forall\bar{D}\in\mathrm{H}^1(\mathbf{W^g}(a,b)_{\frac{m}{2}},\mathbf{W^g}(a,b)_{\frac{m}{2}+1})$, we may assume that

$$\bar{D}(L_m)=0,\quad\bar{D}(I_m)=0,\quad\bar{D}(Y_{m+\frac{1}{2}})=b^{m+1+\frac{1}{2}}Y_{m+1+\frac{1}{2}}.$$

By $\bar{D}([L_m,Y_{n+\frac{1}{2}}])=[\bar{D}(L_m),Y_{n+\frac{1}{2}}]+[L_m,\bar{D}(Y_{n+\frac{1}{2}})]$, we have

$$(n+\frac{1-m+a+bm}{2})b^{m+n+1+\frac{1}{2}}=(n+1+\frac{1-m+a+bm}{2})b^{n+1+\frac{1}{2}}\tag{2.43}$$

Let $m=0$ in (2.43), we get $b^{n+1+\frac{1}{2}}=0$ for all $n\in\mathbb{Z}$. So $\bar{D}(Y_{m+\frac{1}{2}})=0$. Consequently, we get

$$\mathrm{H}^1(\mathbf{W^g}(a,b)_{\frac{m}{2}},\mathbf{W^g}(a,b)_{\frac{m}{2}+1})=0.$$

For $\mathrm{H}^1(\mathbf{W^g}(a,b)_{\frac{m}{2}},\mathbf{W^g}(a,b)_{\frac{m}{2}-1})$, similar to the above discussion, we have the same results. $\square$

Lemma 2.9. *Up to isomorphism,*

$$\mathrm{H}^1(\mathbf{W^g}(a,b)_{\frac{m}{2}},\mathbf{W^g}(a,b)_{\frac{m+1}{2}})=0.$$

Proof. $\forall D\in \mathrm{Der}(\mathbf{W}^{\mathbf{g}}(a,b))_{\frac{1}{2}}$, we assume that

$$D(L_m)=b_1^{m+\frac{1}{2}}Y_{m+\frac{1}{2}},\ D(I_m)=b_2^{m+\frac{1}{2}}Y_{m+\frac{1}{2}},\ D(Y_{m+\frac{1}{2}})=a_{11}^{m+1}L_{m+1}+a_{12}^{m+1}I_{m+1}.$$

By the definition of derivation and the bracket for $\mathbf{W}^{\mathbf{g}}(a,b)$, we have

$$(m-n)b_1^{m+n+\frac{1}{2}}=(m+\frac{1-n+a+bn}{2})b_1^{m+\frac{1}{2}}-(n+\frac{1-m+a+bm}{2})b_1^{n+\frac{1}{2}} \tag{2.44}$$

$$(n+a+bm)b_2^{m+n+\frac{1}{2}}=(n+\frac{1-m+a+bm}{2})b_2^{n+\frac{1}{2}} \tag{2.45}$$

$$(n+\frac{1-m+a+bm}{2})a_{11}^{m+n+1}=(n+1-m)a_{11}^{n+1} \tag{2.46}$$

$$(n+\frac{1-m+a+bm}{2})a_{12}^{m+n+1}=(n-m)b_1^{m+\frac{1}{2}}+(n+1+a+bm)a_{12}^{n+1} \tag{2.47}$$

$$(m-n)b_2^{m+\frac{1}{2}}+(m+a+b(n+1))a_{11}^{n+1}=0 \tag{2.48}$$

$$(m-n)b_2^{m+n+1+\frac{1}{2}}=-(n+\frac{-m+a+b(m+1)}{2})a_{11}^{m+1}+(m+\frac{-n+a+b(n+1)}{2})a_{11}^{n+1} \tag{2.49}$$

Let $n=0$ in (2.44), we have $(1+a)b_1^{m+\frac{1}{2}}=(1-m+a+bm)b_1^{\frac{1}{2}}$. Since $a\notin\mathbb{Z}$ or $a=0,a=1$, we get

$$b_1^{m+\frac{1}{2}}=\frac{1+a-m+bm}{1+a}b_1^{\frac{1}{2}},\quad \forall m\in\mathbb{Z}.$$

Let $m=0$ in (2.45), (2.46) and (2.47), we have

$$(a-1)b_2^{n+\frac{1}{2}}=0, \tag{2.50}$$

$$(a-1)a_{11}^{n+1}=0, \tag{2.51}$$

$$(a+1)a_{12}^{n+1}=-2nb_1^{\frac{1}{2}}. \tag{2.52}$$

Then we get $a_{12}^{n+1}=\dfrac{-2n}{a+1}b_1^{\frac{1}{2}}$, for all $n\in\mathbb{Z}$ since $a\notin\mathbb{Z}$ or $a=0,a=1$. Let $m=-1$ in (2.48) and use (2.51), we obtain

$$(n+1)ba_{11}^{n+1}=(n+1)b_2^{-1+\frac{1}{2}},\quad \forall n\in\mathbb{Z}. \tag{2.53}$$

Case 1: $a\notin\mathbb{Z}$ or $a=0$. It follows from (2.50) and (2.51) that

$$b_2^{n+\frac{1}{2}}=0,\quad a_{11}^{n+1}=0,\quad \forall n\in\mathbb{Z}.$$

So

$$D(L_m)=\frac{1+a-m+bm}{1+a}b_1^{\frac{1}{2}}Y_{m+\frac{1}{2}},\quad D(I_m)=0,\ D(Y_{m+\frac{1}{2}})=\frac{-2m}{1+a}b_1^{\frac{1}{2}}I_{m+1}.$$

Set $E=\dfrac{2b^{\frac{1}{2}}}{1+a}Y_{\frac{1}{2}}$, then

$$D(L_m)=adE(L_m),\quad D(I_m)=adE(I_m),\quad D(Y_{m+\frac{1}{2}})=adE(Y_{m+\frac{1}{2}}).$$

Case 2: $a = 1$. Let $n = -1$ in (2.48), we have

$$b_2^{m+\frac{1}{2}} = -a_{11}^0, \quad m \neq -1. \tag{2.54}$$

By (2.53), we also have

$$b_2^{-1+\frac{1}{2}} = ba_{11}^{n+1}, \quad n \neq -1. \tag{2.55}$$

Let $n = 0, m = 1$ in (2.45) and use (2.54), we get

$$(1 + b)a_{11}^0 = 0. \tag{2.56}$$

On the other hand, let $m + n = -1$ in (2.46), we get

$$a_{11}^{n+1} = \frac{3 - b}{4} a_{11}^0, \quad n \neq -1. \tag{2.57}$$

Subcase 2.1: $b = 0$. From (2.54)-(2.57), we get

$$b_2^{m+\frac{1}{2}} = 0, \quad a_{11}^{n+1} = 0, \quad \forall m, n \in \mathbb{Z}.$$

So we have the same results as case1.
Subcase 2.2: $b \neq 0$. If $b \neq -1$. By (2.56), we have $a_{11}^0 = 0$, then by (2.54), (2.55), (2.57), we obtain

$$b_2^{m+\frac{1}{2}} = 0, \quad a_{11}^{n+1} = 0, \quad \forall m, n \in \mathbb{Z}.$$

So we get the same results as subcase 2.1 completely.
If $b = -1$, By (2.54), (2.55), (2.57), we have

$$a_{11}^{n+1} = a_{11}^0, \quad b_2^{m+\frac{1}{2}} = -a_{11}^0, \quad \forall m, n \in \mathbb{Z}. \tag{2.58}$$

On the other hand, let $n = 0, m = 1$ in (2.49) and use (2.58), we obtain $a_{11}^0 = 0$. Then $a_{11}^{n+1} = b_2^{n+\frac{1}{2}} = 0$, for all $n \in \mathbb{Z}$ and we get the same results as subcase 2.1. $\square$

Lemma 2.10. *Up to isomorphism,*

$$\mathrm{H}^1(\mathbf{W^g}(a,b)_{\frac{m}{2}}, \mathbf{W^g}(a,b)_{\frac{m-1}{2}}) = \begin{cases} \mathbb{C}\bar{D}, & (a,b) = (1,-1) \\ 0, & else \end{cases}$$

where

$$\bar{D}(L_m) = 0,\ \bar{D}(I_m) = 0,\ \bar{D}(Y_{m+\frac{1}{2}}) = I_m.$$

Proof. $\forall D \in \mathrm{Der}(\mathbf{W^g}(a,b))_{-\frac{1}{2}}$, we assume that

$$D(L_m) = b_1^{m-\frac{1}{2}} Y_{m-1+\frac{1}{2}},\ D(I_m) = b_2^{m-\frac{1}{2}} Y_{m-1+\frac{1}{2}},\ D(Y_{m+\frac{1}{2}}) = a_{11}^m L_m + a_{12}^m I_m.$$

By the definition of derivation and the bracket for $\mathbf{W^g}(a,b)$, we have

$$(m-n)b_1^{m+n-\frac{1}{2}} = (m-1+\frac{1-n+a+bn}{2})b_1^{m-\frac{1}{2}} - (n-1+\frac{1-m+a+bm}{2})b_1^{n-\frac{1}{2}} \tag{2.59}$$

$$(n+a+bm)b_2^{m+n-\frac{1}{2}} = (n-1+\frac{1-m+a+bm}{2})b_2^{n-\frac{1}{2}} \tag{2.60}$$

$$(n+\frac{1-m+a+bm}{2})a_{11}^{m+n} = (n-m)a_{11}^{n} \tag{2.61}$$

$$(n+\frac{1-m+a+bm}{2})a_{12}^{m+n} = (n+1-m)b_1^{m-\frac{1}{2}} + (n+a+bm)a_{12}^{n} \tag{2.62}$$

$$(m-1-n)b_2^{m-\frac{1}{2}} + (m+a+bn)a_{11}^{n} = 0 \tag{2.63}$$

$$(m-n)b_2^{m+n+\frac{1}{2}} = -(n+\frac{1-m+a+bm}{2})a_{11}^{m} + (m+\frac{1-n+a+bn}{2})a_{11}^{n} \tag{2.64}$$

Let $n=0$ in (2.59) and $m=0$ in (2.60)-(2.62), we have the following identities

$$(a-1)b_1^{m-\frac{1}{2}} = (-1-m+a+bm)b_1^{-\frac{1}{2}}, \tag{2.65}$$

$$(a-1)a_{12}^{n} = -2(n+1)b_1^{-\frac{1}{2}}, \tag{2.66}$$

$$(a+1)b_2^{n-\frac{1}{2}} = 0, \quad (a+1)a_{11}^{n} = 0.$$

Since $a \notin \mathbb{Z}$ or $a=0$ or $a=1$, we get

$$b_2^{n-\frac{1}{2}} = 0, \quad a_{11}^{n} = 0, \quad \forall n \in \mathbb{Z}.$$

Case 1: $a \notin \mathbb{Z}$ or $a=0$. By (2.65) and (2.66), we get

$$b_1^{m-\frac{1}{2}} = \frac{-1+a-m+bm}{a-1}b_1^{-\frac{1}{2}}, \quad a_{12}^{n} = \frac{-2(n+1)}{a-1}b_1^{-\frac{1}{2}}, \quad \forall m,n \in \mathbb{Z}.$$

Then

$$D(L_m) = \frac{-1+a-m+bm}{a-1}b_1^{-\frac{1}{2}}Y_{m-\frac{1}{2}}, \quad D(I_m)=0, \quad D(Y_{m+\frac{1}{2}}) = \frac{-2(m+1)}{a-1}b_1^{-\frac{1}{2}}I_m.$$

Set $E = \frac{2b_1^{-\frac{1}{2}}}{a-1}Y_{-\frac{1}{2}}$, so

$$D(L_m) = adE(L_m), \quad D(I_m) = adE(I_m), \quad D(Y_{m+\frac{1}{2}}) = adE(Y_{m+\frac{1}{2}}).$$

Case 2: $a=1$. By (2.66), we have $b_1^{-\frac{1}{2}}=0$. So let $m+n=0$ in (2.59), we have

$$(3-b)(b_1^{m-\frac{1}{2}} + b_1^{-m-\frac{1}{2}}) = 0,$$

then

$$(b_1^{m-\frac{1}{2}} + b_1^{-m-\frac{1}{2}}) = 0, \quad b \neq 3, \forall m \in \mathbb{Z}.$$

On the other hand, let $n=1$ in (2.59), we have

$$(m-1)b_1^{m+1-\frac{1}{2}} = (m+\frac{b-1}{2})b_1^{m-\frac{1}{2}} - (1+\frac{b-1}{2}m)b_1^{1-\frac{1}{2}},$$

that is

$$(m-1)(b_1^{m+1-\frac{1}{2}} - (m+1)b_1^{1-\frac{1}{2}}) = (m+\frac{b-1}{2})(b_1^{m-\frac{1}{2}} - mb_1^{1-\frac{1}{2}}) \tag{2.67}$$

Induction on $m \in \mathbb{Z}, m \geq 3$ in (2.67), we get

$$b_1^{m-\frac{1}{2}} = \frac{(3+b)(5+b)\cdots(2m-3+b)}{2^{m-2}(m-2)!}(b_1^{2-\frac{1}{2}} - 2b_1^{1-\frac{1}{2}}) + mb_1^{1-\frac{1}{2}}. \tag{2.68}$$

Let $n = -1$ in (2.62), we have

$$\frac{b-1}{2} m a_{12}^{m-1} = bma_{12}^{-1} - mb_1^{m-\frac{1}{2}}. \tag{2.69}$$

Subcase 2.1: $b = 1$. By (2.67) and using induction, we have

$$b_1^{m-\frac{1}{2}} = (m-1)b_1^{2-\frac{1}{2}} - (m-2)b_1^{1-\frac{1}{2}}, \quad \forall m \in \mathbb{Z}, m \neq 0. \tag{2.70}$$

On the other hand, by (2.69), we get

$$b_1^{m-\frac{1}{2}} = a_{12}^{-1}, \ \forall m \in \mathbb{Z}, m \neq 0. \tag{2.71}$$

Let $n = 0$ in (2.62) and use (2.71), we obtain

$$a_{12}^m = (m+1)a_{12}^0 - (m-1)a_{12}^{-1}, \ m \neq 0. \tag{2.72}$$

Let $m = -1$ in (2.72), we get $a_{12}^{-1} = 0$. Consequently, we obtain

$$b_1^{m-\frac{1}{2}} = 0, \quad a_{12}^m = (m+1)a_{12}^0, \quad \forall m \in \mathbb{Z}.$$

So

$$D(L_m) = 0, \quad D(I_m) = 0, \quad D(Y_{m+\frac{1}{2}}) = (m+1)a_{12}^0 I_m.$$

Set $E = -a_{12}^0 Y_{-\frac{1}{2}}$, then we have

$$D(L_m) = adE(L_m), \ D(I_m) = adE(I_m), \ D(Y_{m+\frac{1}{2}}) = adE(Y_{m+\frac{1}{2}}).$$

Subcase 2.2: $b \neq 1$. By (2.69), we have

$$a_{12}^{m-1} = \frac{2b}{b-1}a_{12}^{-1} - \frac{2}{b-1}b_1^{m-\frac{1}{2}}, \quad m \neq 0. \tag{2.73}$$

Let $n + 1 - m = 0$ in (2.62), we have

$$(n+1)(b+1)a_{12}^{2n+1} = 2(n+1)(b+1)a_{12}^n,$$

then

$$(b+1)a_{12}^{2n+1} = 2(b+1)a_{12}^n, \quad n \neq -1. \tag{2.74}$$

If $b \neq -1$, we have $a_{12}^{2n+1} = 2a_{12}^n, \ n \neq -1$. using (2.73), we get

$$b_1^{2m+2-\frac{1}{2}} = 2b_1^{m+1-\frac{1}{2}} - ba_{12}^{-1}, \ m \neq -1. \tag{2.75}$$

Especially, $b_1^{2-\frac{1}{2}} - 2b_1^{1-\frac{1}{2}} = -ba_{12}^{-1}$. Combine (2.75) and (2.68), we get

$$ba_{12}^{-1} = 0. \tag{2.76}$$

Consequently, we get

$$b_1^{m-\frac{1}{2}} = mb_1^{1-\frac{1}{2}}, \quad a_{12}^{m-1} = -\frac{2m}{b-1}b_1^{1-\frac{1}{2}}, \quad \forall m \in \mathbb{Z}.$$

Set $E = \frac{2}{b-1}b_1^{1-\frac{1}{2}}Y_{-\frac{1}{2}}$, then we have

$$D(L_m) = adE(L_m),\ D(I_m) = adE(I_m),\ D(Y_{m+\frac{1}{2}}) = adE(Y_{m+\frac{1}{2}}).$$

If $b = -1$, by (2.68), we still have

$$b_1^{m-\frac{1}{2}} = mb_1^{1-\frac{1}{2}}, \quad a_{12}^{m-1} = a_{12}^{-1} + mb_1^{1-\frac{1}{2}}, \forall m \in \mathbb{Z}.$$

Set $E = -b_1^{1-\frac{1}{2}}Y_{-\frac{1}{2}}$, then we have

$$D(L_m) = adE(L_m),\ D(I_m) = adE(I_m),\ D(Y_{m+\frac{1}{2}}) = adE(Y_{m+\frac{1}{2}}) + a_{12}^{-1}I_m.$$

Then we get

$$\mathrm{H}^1(\mathbf{W}^{\mathbf{g}}(a,b)_{\frac{m}{2}}, \mathbf{W}^{\mathbf{g}}(a,b)_{\frac{m-1}{2}}) = \mathbb{C}\bar{D},$$

Where $\bar{D}(L_m) = 0, \bar{D}(I_m) = 0, \bar{D}(Y_{m+\frac{1}{2}}) = I_m$. $\square$

Now, by lemma 2.7-2.10, we obtain the main theorem of this section as following.

Theorem 2.11. *Up to isomorphism, we have*

$$\mathrm{H}^1(\mathbf{W}^{\mathbf{g}}(a,b), \mathbf{W}^{\mathbf{g}}(a,b)) = \begin{cases} \mathbb{C}D_1 \bigoplus \mathbb{C}D_2^{0,0} \bigoplus \mathbb{C}D_3, & (a,b) = (0,0), \\ \mathbb{C}D_1 \bigoplus \mathbb{C}D_2^{0,1}, & (a,b) = (0,1), \\ \mathbb{C}D_1 \bigoplus \mathbb{C}D_2^{0,2}, & (a,b) = (0,2), \\ \mathbb{C}D_3^{1,-1}, & (a,b) = (1,-1), \\ \mathbb{C}D_1, & otherwise, \end{cases}$$

where for all $m \in \mathbb{Z}$,

$$D_1(L_m) = 0, \qquad D_1(I_m) = I_m, \qquad D_1(Y_{m+\frac{1}{2}}) = Y_{m+\frac{1}{2}}, \tag{2.77}$$

$$D_2^{0,0}(L_m) = (m-1)I_m, \qquad D_2^{0,0}(I_m) = 0, \qquad D_2^{0,0}(Y_{m+\frac{1}{2}}) = 0, \tag{2.78}$$

$$D_2^{0,1}(L_m) = (m^2-m)I_m, \qquad D_2^{0,1}(I_m) = 0, \qquad D_2^{0,1}(Y_{m+\frac{1}{2}}) = 0, \tag{2.79}$$

$$D_2^{0,2}(L_m) = m^3 I_m, \qquad D_2^{0,2}(I_m) = 0, \qquad D_2^{0,2}(Y_{m+\frac{1}{2}}) = 0, \tag{2.80}$$

$$D_3(L_m) = mI_m, \qquad D_3(I_m) = 0, \qquad D_3(Y_{m+\frac{1}{2}}) = 0, \tag{2.81}$$

$$D_3^{1,-1}(L_m) = 0, \qquad D_3^{1,-1}(I_m) = 0, \qquad D_3^{1,-1}(Y_{m+\frac{1}{2}}) = I_m. \tag{2.82}$$

3. **Automorphism groups of $\mathbf{W^g}(a,b)$**

Denote by $\mathfrak{I}$ the inner automorphism group of $\mathbf{W^g}(a,b)$. Obviously, $\mathfrak{I}$ is generated by $\{\exp k_i\,\mathrm{ad}\,I_i, \exp l_i\,\mathrm{ad}\,Y_i \mid k_i, l_i \in \mathbb{C},\, i,j \in \mathbb{Z}\}$ and is a normal subgroup of $\mathrm{Aut}(\mathbf{W^g}(a,b))$. We can verify easily that $\mathfrak{I}$ is isomorphism to $\mathbb{C}^\infty \times \mathbb{C}^\infty$ as sets. But by computation, these generators satisfy the following relation:

$$(\exp\alpha\,\mathrm{ad}\,Y_{j+\frac{1}{2}})(\exp\beta\,\mathrm{ad}\,Y_{i+\frac{1}{2}})(\exp\alpha\,\mathrm{ad}\,Y_{j+\frac{1}{2}})^{-1}(\exp\beta\,\mathrm{ad}\,Y_{i+\frac{1}{2}})^{-1} = \exp\gamma\,\mathrm{ad}\,I_{i+j+1},$$

where $\gamma = \alpha\beta(j-i)$. And the others is commutable each other. Hence $\mathfrak{I}$ is isomorphism to $\mathbb{C}^\infty \rtimes \mathbb{C}^\infty$ as groups.

For any $\prod_{j=s}^{t}\exp(k_{i_j}\mathrm{ad}I_{i_j} + l_{i_j}\mathrm{ad}Y_{i_j+\frac{1}{2}}) \in \mathfrak{I}$, we have

$$\prod_{j=s}^{t}\exp(k_{i_j}\mathrm{ad}I_{i_j} + l_{i_j}\mathrm{ad}Y_{i_j+\frac{1}{2}})(I_n) = I_n, \tag{3.1}$$

$$\prod_{j=s}^{t}\exp(k_{i_j}\mathrm{ad}I_{i_j} + l_{i_j}\mathrm{ad}Y_{i_j+\frac{1}{2}})(Y_{n+\frac{1}{2}}) = Y_{n+\frac{1}{2}} + \sum_{j=s}^{t} l_{i_j}(i_j - n)I_{i_j+n+1}. \tag{3.2}$$

As $\mathbf{I^g}(a,b)$ is an unique maximal proper ideal of $\mathbf{W^g}(a,b)$, we have the following lemma.

Lemma 3.1. *For any $\sigma \in \mathrm{Aut}(\mathbf{W^g}(a,b))$, $\sigma(I_n)$, $\sigma(Y_{n+\frac{1}{2}}) \in \mathbf{I^g}(a,b)$ for all $n \in \mathbb{Z}$.*

□

For any $\sigma \in \mathrm{Aut}(\mathbf{W^g}(a,b))$, denote $\sigma|_{\mathbf{W}} = \sigma'$. Then according to the automorphisms of the classical Witt algebra, we have $\sigma'(L_m) = \epsilon\alpha^m L_{\epsilon m}$ for all $m \in \mathbb{Z}$, where $\alpha \in \mathbb{C}^*$ and $\epsilon \in \{\pm 1\}$. So we have the following lemmas.

Lemma 3.2. *For $a \notin \mathbb{Z}$,* $\mathrm{Aut}(\mathbf{W^g}(a,b)) \cong (\mathbb{C}^\infty \rtimes \mathbb{C}^\infty) \rtimes (\mathbb{C}^* \times \mathbb{C}^*)$.
where $\mathbb{C}^\infty = \{(a_i)_{i\in\mathbb{Z}} \mid a_i \in \mathbb{C},$ all but a finite number of the a_i are zero $\}$, $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$.

Proof. For any $\sigma \in \mathrm{Aut}(\mathbf{W^g}(a,b))$, assume

$$\sigma(L_0) = \epsilon L_0 + \sum_{i=p}^{q}\lambda_i I_i + \sum_{j=s}^{t} l_j Y_{j+\frac{1}{2}},$$

where $\lambda_i,\ l_j \in \mathbb{C}$, $p,q,s,t \in \mathbb{Z}$, $p \le i \le q$, $s \le j \le t$.
Since $a \notin \mathbb{Z}$, we let $\tau_1 = \prod_{i=p}^{q}\exp(\frac{\lambda_i}{\epsilon(i+a)}\mathrm{ad}I_i)\prod_{j=s}^{t}\exp(\frac{l_j}{\epsilon(j+\frac{1+a}{2})}\mathrm{ad}Y_{j+\frac{1}{2}}) \in \mathfrak{I}$, then

$$\tau_1(\epsilon L_0) = \epsilon L_0 + \sum_{i=p}^{q}\lambda_i I_i + \sum_{j=s}^{t} l_j Y_{j+\frac{1}{2}} + \sum_{k=p'}^{q'} b_k I_k = \sigma(L_0) + \sum_{k=p'}^{q'} b_k I_k,$$

where $b_k \in \mathbb{C}$, $p', q' \in \mathbb{Z}$. On the other hand, let $\tau_2 = \prod_{k=p'}^{q'} \exp(\frac{-b_k}{\epsilon(k+a)} \mathrm{ad} I_k) \in \mathfrak{I}$ and $\tau = \tau_2\tau_1 \in \mathfrak{I}$, we have $\tau(\epsilon L_0) = \sigma(L_0)$. Consequently, $\bar{\sigma} = \tau^{-1}\sigma$ and $\bar{\sigma}(L_0) = \epsilon L_0$.
By Lemma 3.1, we may assume

$$\bar{\sigma}(L_n) = \alpha^n \epsilon L_{\epsilon n} + \sum \lambda_{n_i} I_{n_i} + \sum \mu_{m_j} Y_{m_j+\frac{1}{2}}, \quad n \neq 0,$$

$$\bar{\sigma}(I_m) = \sum c_p I_p + \sum d_q Y_{q+\frac{1}{2}},$$

$$\bar{\sigma}(Y_{m+\frac{1}{2}}) = \sum e_u I_u + \sum f_v Y_{v+\frac{1}{2}},$$

where each formula is of finite terms and $\lambda_{n_i}, \mu_{m_j}, c_p, d_q, e_u, f_v \in \mathbb{C}, n_i, m_j, p, q, u, v \in \mathbb{Z}$, $\alpha \in \mathbb{C}^*, \epsilon = \pm 1$. For any $n \neq 0$, by the relation $[\bar{\sigma}(L_0), \bar{\sigma}(L_n)] = -n\bar{\sigma}(L_n)$, we have

$$\lambda_{n_i}[\epsilon(n_i + a) - n] = 0, \ \mu_{m_j}[\epsilon(m_j + \frac{1+a}{2}) - n] = 0.$$

Since $a \notin Z$, this forces that $\lambda_{n_i} = 0$, $\mu_{m_j} = 0$ for all $n_i, m_j \in \mathbb{Z}$. So we obtain

$$\bar{\sigma}(L_n) = \alpha^n \epsilon L_{\epsilon n}, \ \forall n \in Z.$$

Since $[\bar{\sigma}(L_0), \bar{\sigma}(I_m)] = -(m + a)\bar{\sigma}(I_m), [\bar{\sigma}(L_0), \bar{\sigma}(Y_{m+\frac{1}{2}})] = -(m + \frac{1+a}{2})\bar{\sigma}(Y_{m+\frac{1}{2}})$, we have

$$c_p[\epsilon(p + a) - (m + a)] = 0, \quad d_q[\epsilon(q + \frac{1+a}{2}) - (m + a)] = 0,$$

$$e_u[\epsilon(u + a) - (m + \frac{1+a}{2})] = 0, \quad f_v[\epsilon(v + \frac{1+a}{2}) - (m + \frac{1+a}{2})] = 0.$$

Therefore, if $\epsilon = 1$, then $p = m, v = m$ and all $d_q = 0, e_u = 0$; If $\epsilon = -1$, then $p = -m - 2a$ and $q = -m - \dfrac{1+3a}{2}$, which implies that $a \in \dfrac{\mathbb{Z}}{2}$ and $a \in \dfrac{2\mathbb{Z}-1}{3}$. We also have all $f_v = 0$ and $u = -m - \dfrac{1+3a}{2}$ which implies that $a \in \dfrac{2\mathbb{Z}-1}{3}$ if $\epsilon = -1$. Since $a \notin \mathbb{Z}$, either $a \in \dfrac{\mathbb{Z}}{2}$ or $a \in \dfrac{2\mathbb{Z}-1}{3}$ holds.
If $a \in \dfrac{\mathbb{Z}}{2}$, from the above discussion, it forces that $\epsilon = 1$, otherwise it must be that $\bar{\sigma}(Y_{m+\frac{1}{2}}) = 0$, which is contradiction. If $a \in \dfrac{2\mathbb{Z}-1}{3}$, we have

$$\bar{\sigma}(I_m) = d_{-m-\frac{1+3a}{2}} Y_{-m-\frac{1+3a}{2}+\frac{1}{2}}, \qquad \bar{\sigma}(Y_{m+\frac{1}{2}}) = e_{-m-\frac{1+3a}{2}} I_{-m-\frac{1+3a}{2}},$$

where $d_{-m-\frac{1+3a}{2}}, e_{-m-\frac{1+3a}{2}} \in \mathbb{C}^*$. But by $[\bar{\sigma}(Y_{m+\frac{1}{2}}), \bar{\sigma}(Y_{n+\frac{1}{2}})] = (m-n)\bar{\sigma}(I_{m+n+1})$, we have $d_{-m-n-1-\frac{1+3a}{2}} = 0$ which means $\bar{\sigma}(I_m) = 0$. This also is a contradiction. So we obtain

$$\bar{\sigma}(L_m) = \alpha^m L_m, \quad \bar{\sigma}(I_m) = c_m I_m, \quad \bar{\sigma}(Y_{m+\frac{1}{2}}) = f_m Y_{m+\frac{1}{2}}, \quad \forall m \in \mathbb{Z}.$$

where $\alpha, c_m, f_m \in \mathbb{C}^*$.
On the other hand, from the relations that

$$[\bar{\sigma}(L_m), \bar{\sigma}(I_n)] = -(n+a+bm)\bar{\sigma}(I_{m+n}),$$
$$[\bar{\sigma}(L_m), \bar{\sigma}(Y_{n+\frac{1}{2}})] = -(n+\frac{1-m+a+bm}{2})\bar{\sigma}(Y_{m+n+\frac{1}{2}}),$$
$$[\bar{\sigma}(Y_{m+\frac{1}{2}}), \bar{\sigma}(Y_{n+\frac{1}{2}})] = (m-n)\bar{\sigma}(I_{m+n+1}),$$

we have

$$(n+a+bm)(C_{m+n}-C_n\alpha^m)=0, \quad \alpha^m f_n = f_{m+n}, \quad f_m f_n = C_{m+n+1},$$

for all $m, n \in \mathbb{Z}$. It is easy to deduce that

$$C_m = \alpha^m \mu, \quad f_m = \alpha^m \sqrt{\alpha\mu}, \quad \forall m \in \mathbb{Z},$$

where μ is a nonzero complex number. Therefore,

$$\bar{\sigma}(L_m) = \alpha^m L_m, \quad \bar{\sigma}(I_m) = \alpha^m \mu I_m, \quad \bar{\sigma}(Y_{m+\frac{1}{2}}) = \alpha^m \sqrt{\alpha\mu} Y_{m+\frac{1}{2}} \quad \forall\, m \in \mathbb{Z}. \tag{3.3}$$

Conversely, if $\bar{\sigma}$ is a linear operator on $\mathbf{W}^{\mathfrak{g}}(a,b)$ satisfying (3.3) for some $\alpha, \mu \in \mathbb{C}^*$, then it is easy to check that $\bar{\sigma} \in \mathrm{Aut}(\mathbf{W}^{\mathfrak{g}}(a,b))$. Denote by $\bar{\sigma}(\alpha,\mu)$ the automorphism of $\mathbf{W}^{\mathfrak{g}}(a,b)$ satisfying (3.3), then

$$\bar{\sigma}(\alpha_1,\mu_1)\bar{\sigma}(\alpha_2,\mu_2) = \bar{\sigma}(\alpha_1\alpha_2, \mu_1\mu_2), \tag{3.4}$$

and $\bar{\sigma}(\alpha_1,\mu_1) = \bar{\sigma}(\alpha_2,\mu_2)$ if and only if $\alpha_1 = \alpha_2, \mu_1 = \mu_2$. Let

$$\mathfrak{a}_1 = \{\bar{\sigma}_{\alpha,\mu} \mid \alpha, \mu \in \mathbb{C}^*\}.$$

By (3.4), $\mathfrak{a}_1 \cong \mathbb{C}^* \times \mathbb{C}^*$ is a subgroup of $\mathrm{Aut}(\mathbf{W}^{\mathfrak{g}}(a,b))$. On the other hand, similar to the proof in [7], [8] or [22], $\mathfrak{I} \cong \mathbb{C}^\infty \rtimes \mathbb{C}^\infty$. Consequently, we have

$$\mathrm{Aut}(\mathbf{W}^{\mathfrak{g}}(a,b)) = \mathfrak{I} \rtimes \mathfrak{a}_1 \cong \mathfrak{I} \rtimes (\mathbb{C}^* \times \mathbb{C}^*),$$

where $a \notin \mathbb{Z}$. □

Lemma 3.3. *For $a=0$,* $\mathrm{Aut}(\mathbf{W}^{\mathfrak{g}}(a,b)) \cong (\mathbb{C}^\infty \rtimes \mathbb{C}^\infty) \rtimes (\mathbb{Z}_2 \ltimes (\mathbb{C}^* \times \mathbb{C}^*))$. *where* $\mathbb{C}^\infty = \{(a_i)_{i\in\mathbb{Z}} \mid a_i \in \mathbb{C},$ all but a finite number of the a_i are zero $\}$, $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$.

Proof. For any $\sigma \in \mathrm{Aut}(\mathbf{W}^{\mathfrak{g}}(a,b))$, $\sigma(L_0)$ is the same as in lemma 3.2. Since $a=0$. Let $\tau_1 = \prod_{i=p}^{q} \exp(\frac{\lambda_i}{\epsilon i}\mathrm{ad}I_i) \prod_{j=s}^{t} \exp(\frac{l_j}{\epsilon(j+\frac{1}{2})}\mathrm{ad}Y_{j+\frac{1}{2}}) \in \mathfrak{I}$, where $i \neq 0$. We use the same idea as in Lemma 3.2, then there exists $\tau \in \mathfrak{I}$ such that $\sigma(L_0) = \tau(\epsilon L_0) + \lambda_0 I_0$. Set $\bar{\sigma} = \tau^{-1}\sigma$, then $\bar{\sigma}(L_0) = \epsilon L_0 + \lambda_0 I_0$. As the assumption in Lemma 3.2, for any $n \neq 0, m \in \mathbb{Z}$, since

$$[\bar{\sigma}(L_0), \bar{\sigma}(L_n)] = -n\bar{\sigma}(L_n), \quad [\bar{\sigma}(L_0), \bar{\sigma}(I_m)] = -m\bar{\sigma}(I_m)$$

we have

$$\mu_{m_j}[\epsilon(m_j+\frac{1}{2})-n]=0,\quad d_q[\epsilon(q+\frac{1}{2})-n]=0.$$

This forces that $\mu_{m_j}=0, d_q=0$ for all $m_j, q\in\mathbb{Z}$. Therefore,

$$\bar{\sigma}(\mathbf{W}(a,b))=\mathbf{W}(a,b).$$

On the other hand, according to $[\bar{\sigma}(L_0),\bar{\sigma}(Y_{m+\frac{1}{2}})]=-(m+\frac{1}{2})\bar{\sigma}(Y_{m+\frac{1}{2}})$, we have

$$e_u(\epsilon u-m-\frac{1}{2})=0,\quad f_v[\epsilon(v+\frac{1}{2})-(m+\frac{1}{2})]=0.$$

So $e_u=0, v=\epsilon(m+\frac{1}{2})-\frac{1}{2}$ and

$$\bar{\sigma}(Y_{m+\frac{1}{2}})=f_{\epsilon(m+\frac{1}{2})-\frac{1}{2}}Y_{\epsilon(m+\frac{1}{2})},$$

for all $m\in\mathbb{Z}$. Consequently, by [21], we have the following cases.

Case 1 $b=0$. Assume

$$\bar{\sigma}(L_m)=\epsilon\alpha^m L_{\epsilon m}+\alpha^m(cm+d)I_{\epsilon m},\quad \bar{\sigma}(I_m)=\alpha^m\mu I_{\epsilon m},$$

$$\bar{\sigma}(Y_{m+\frac{1}{2}})=f_{\epsilon(m+\frac{1}{2})-\frac{1}{2}}Y_{\epsilon(m+\frac{1}{2})}\qquad\forall\, m\in\mathbb{Z},$$

where $\alpha,\mu,f_{\epsilon(m+\frac{1}{2})-\frac{1}{2}}\in\mathbb{C}^*, c,d\in\mathbb{C}$. By

$$[\bar{\sigma}(L_m),\bar{\sigma}(Y_{n+\frac{1}{2}})]=-(n+\frac{1-m}{2})\bar{\sigma}(Y_{m+n+\frac{1}{2}}),\;\;[\bar{\sigma}(Y_{m+\frac{1}{2}}),\bar{\sigma}(Y_{n+\frac{1}{2}})]=(m-n)\bar{\sigma}(I_{m+n+1}),$$

we have

$$f_{\epsilon(m+\frac{1}{2})-\frac{1}{2}}=\alpha^m f_{\epsilon(0+\frac{1}{2})-\frac{1}{2}},\quad \epsilon f_{\epsilon(m+\frac{1}{2})-\frac{1}{2}}f_{\epsilon(n+\frac{1}{2})-\frac{1}{2}}=\alpha\mu.$$

It is easy to deduce that

$$f_{\epsilon(m+\frac{1}{2})-\frac{1}{2}}=\alpha^m\sqrt{\epsilon\alpha\mu}.$$

Therefore

$$\bar{\sigma}(L_m)=\epsilon\alpha^m L_{\epsilon m}+\alpha^m(cm+d)I_{\epsilon m},\quad \bar{\sigma}(I_m)=\alpha^m\mu I_{\epsilon m},\tag{3.5}$$

$$\bar{\sigma}(Y_{m+\frac{1}{2}})=\alpha^m\sqrt{\epsilon\alpha\mu}Y_{\epsilon(m+\frac{1}{2})}\qquad\forall\, m\in\mathbb{Z}.\tag{3.6}$$

Denote by $\bar{\sigma}(\epsilon,\alpha,\mu,c,d)$ the automorphism of $\mathbf{W}^g(0,0)$ satisfying (3.5) and (3.6), then

$$\bar{\sigma}(\epsilon_1,\alpha_1,\mu_1,c_1,d_1)\bar{\sigma}(\epsilon_2,\alpha_2,\mu_2,c_2,d_2)=\bar{\sigma}(\epsilon_1\epsilon_2,\alpha_1^{\epsilon_2}\alpha_2,\mu_1\mu_2,c_1+\mu_1c_2,\epsilon_2d_1+\mu_1d_2),\tag{3.7}$$

and $\bar{\sigma}(\epsilon_1,\alpha_1,\mu_1,c_1,d_1)=\bar{\sigma}(\epsilon_2,\alpha_2,\mu_2,c_2,d_2)$ if and only if $\epsilon_1=\epsilon_2$, $\alpha_1=\alpha_2$, $\mu_1=\mu_2$, $c_1=c_2$, $d_1=d_2$. Let

$$\bar{\sigma}_\epsilon=\bar{\sigma}(\epsilon,1,1,0,0),\quad \bar{\tau}_{\alpha,\mu}=\bar{\sigma}(1,\alpha,\mu,0,0),\quad \bar{\sigma}_{c,d}=\bar{\sigma}(1,1,1,c,d)$$

$$\mathfrak{a}_3=\{\bar{\sigma}_\epsilon\mid\epsilon\in\{\pm1\}\},\quad \mathfrak{b}_3=\{\bar{\tau}_{\alpha,\mu}\mid\alpha,\mu\in\mathbb{C}^*\},\quad \mathfrak{c}_3=\{\bar{\sigma}_{c,d}\mid c,d\in\mathbb{C}\}.$$

From (3.7) we can see that $\mathfrak{a}_3$, $\mathfrak{b}_3$ and $\mathfrak{c}_3$ are all subgroups of $\mathrm{Aut}(\mathbf{W}^g(0,0))$, and $\mathfrak{a}_3 \cong \mathbb{Z}_2, \mathfrak{b}_3 \cong \mathbb{C}^* \times \mathbb{C}^*, \mathfrak{c}_3 \cong \mathbb{C} \times \mathbb{C}$. Using (3.1), (3.2) and (3.7), one can deduce that $\mathfrak{I}$ commutates with $\mathfrak{c}_3$. By (3.7), we have the following relations:

$$\bar{\sigma}_\epsilon \bar{\tau}_{\alpha,\mu} = \bar{\sigma}(\epsilon, \alpha, \mu, 0, 0), \quad \bar{\tau}_{\alpha,\mu}\bar{\sigma}_{c,d} = \bar{\sigma}(1, \alpha, \mu, \mu c, \mu d), \quad \bar{\sigma}_\epsilon \bar{\sigma}_{c,d} = \bar{\sigma}(\epsilon, 1, 1, c, d),$$

$$\bar{\tau}_{\alpha,\mu}\bar{\sigma}_\epsilon = \bar{\sigma}(\epsilon, \alpha^\epsilon, \mu, 0, 0), \quad \bar{\sigma}_{c,d}\bar{\tau}_{\alpha,\mu} = \bar{\sigma}(1, \alpha, \mu, c, d), \quad \bar{\sigma}_{c,d}\bar{\sigma}_\epsilon = \bar{\sigma}(\epsilon, 1, 1, c, \epsilon d),$$

$$\bar{\tau}^{-1}_{\alpha,\mu}\bar{\sigma}_\epsilon\bar{\tau}_{\alpha,\mu} = \bar{\sigma}(\epsilon, \alpha^{1-\epsilon}, 1, 0, 0), \quad \bar{\sigma}_\epsilon\bar{\tau}_{\alpha,\mu}\bar{\sigma}_\epsilon = \bar{\tau}_{\alpha^\epsilon,\mu},$$

$$\bar{\sigma}^{-1}_{c,d}\bar{\tau}_{\alpha,\mu}\bar{\sigma}_{c,d} = \bar{\sigma}(1, \alpha, \mu, (\mu-1)c, (\mu-1)d), \quad \bar{\tau}^{-1}_{\alpha,\mu}\bar{\sigma}_{c,d}\bar{\tau}_{\alpha,\mu} = \bar{\sigma}_{\mu^{-1}c,\mu^{-1}d},$$

$$\bar{\sigma}^{-1}_{c,d}\bar{\sigma}_\epsilon\bar{\sigma}_{c,d} = \bar{\sigma}(\epsilon, 1, 1, 0, (1-\epsilon)d), \quad \bar{\sigma}_\epsilon\bar{\sigma}_{c,d}\bar{\sigma}_\epsilon = \bar{\sigma}_{c,\epsilon d}.$$

Therefore, $\mathfrak{I}\mathfrak{c}_3$ is an abelian normal subgroup of $\mathrm{Aut}(\mathbf{W}^g(0,0))$. Similar to [22], we have

$$\mathrm{Aut}(\mathbf{W}(0,0)) = (\mathfrak{I}\mathfrak{c}_3) \rtimes (\mathfrak{a}_3 \ltimes \mathfrak{b}_3) \cong (\mathbb{C}^\infty \rtimes \mathbb{C}^\infty) \rtimes (\mathbb{Z}_2 \ltimes (\mathbb{C}^* \times \mathbb{C}^*)).$$

Case 2 $b = 1$. According to [21], we may assume

$$\bar{\sigma}(L_m) = \epsilon\alpha^m L_{\epsilon m} + \alpha^m m[mc + d] I_{\epsilon m}, \quad \bar{\sigma}(I_m) = \alpha^m \mu I_{\epsilon m},$$

$$\bar{\sigma}(Y_{m+\frac{1}{2}}) = f_{\epsilon(m+\frac{1}{2})-\frac{1}{2}} Y_{\epsilon(m+\frac{1}{2})}, \ \forall\, m \in \mathbb{Z}.$$

where $\alpha, \mu, f_{\epsilon(m+\frac{1}{2})-\frac{1}{2}} \in \mathbb{C}^*, c, d \in \mathbb{C}$. Similarly, we can deduce that

$$f_{\epsilon(m+\frac{1}{2})-\frac{1}{2}} = \alpha^m \sqrt{\epsilon\alpha\mu}.$$

Therefore

$$\bar{\sigma}(L_m) = \epsilon\alpha^m L_{\epsilon m} + \alpha^m m(cm + d) I_{\epsilon m}, \quad \bar{\sigma}(I_m) = \alpha^m \mu I_{\epsilon m}, \tag{3.8}$$

$$\bar{\sigma}(Y_{m+\frac{1}{2}}) = \alpha^m \sqrt{\epsilon\alpha\mu}\ Y_{\epsilon(m+\frac{1}{2})} \qquad \forall\, m \in \mathbb{Z}. \tag{3.9}$$

Denote by $\bar{\sigma}(\epsilon, \alpha, \mu, c, d)$ the automorphism of $\mathbf{W}^g(0,1)$ satisfying (3.8) and (3.9), then

$$\bar{\sigma}(\epsilon_1, \alpha_1, \mu_1, c_1, d_1)\bar{\sigma}(\epsilon_2, \alpha_2, \mu_2, c_2, d_2) = \bar{\sigma}(\epsilon_1\epsilon_2, \alpha_1^{\epsilon_2}\alpha_2, \mu_1\mu_2, \epsilon_2 c_1 + \mu_1 c_2, d_1 + \mu_1 d_2), \tag{3.10}$$

and $\bar{\sigma}(\epsilon_1, \alpha_1, \mu_1, c_1, d_1) = \bar{\sigma}(\epsilon_2, \alpha_2, \mu_2, c_2, d_2)$ if and only if $\epsilon_1 = \epsilon_2$, $\alpha_1 = \alpha_2$, $\mu_1 = \mu_2$, $c_1 = c_2$, $d_1 = d_2$. Let

$$\bar{\sigma}_\epsilon = \bar{\sigma}(\epsilon, 1, 1, 0, 0), \quad \bar{\tau}_{\alpha,\mu} = \bar{\sigma}(1, \alpha, \mu, 0, 0), \quad \bar{\sigma}_{c,d} = \bar{\sigma}(1, 1, 1, c, d)$$

$$\mathfrak{a}_4 = \{\bar{\sigma}_\epsilon \mid \epsilon \in \{\pm 1\}\}, \quad \mathfrak{b}_4 = \{\bar{\tau}_{\alpha,\mu} \mid \alpha, \mu \in \mathbb{C}^*\}, \quad \mathfrak{c}_4 = \{\bar{\sigma}_{c,d} \mid c, d \in \mathbb{C}\}.$$

From (3.10) we can see that $\mathfrak{a}_4$, $\mathfrak{b}_4$ and $\mathfrak{c}_4$ are all subgroups of $\mathrm{Aut}(\mathbf{W}^g(0,1))$, and $\mathfrak{a}_4 \cong \mathbb{Z}_2, \mathfrak{b}_4 \cong \mathbb{C}^* \times \mathbb{C}^*, \mathfrak{c}_4 \cong \mathbb{C} \times \mathbb{C}$. Similar to the proof above, using (3.1), (3.2) and (3.10), we also have $\mathfrak{I}$ commutates with $\mathfrak{c}_4$. By (3.10), we have the following relations:

$$\bar{\sigma}_\epsilon \bar{\tau}_{\alpha,\mu} = \bar{\sigma}(\epsilon, \alpha, \mu, 0, 0), \quad \bar{\tau}_{\alpha,\mu}\bar{\sigma}_{c,d} = \bar{\sigma}(1, \alpha, \mu, \mu c, \mu d), \quad \bar{\sigma}_\epsilon \bar{\sigma}_{c,d} = \bar{\sigma}(\epsilon, 1, 1, c, d),$$

$$\bar\tau_{\alpha,\mu}\bar\sigma_\epsilon=\bar\sigma(\epsilon,\alpha^\epsilon,\mu,0,0),\quad \bar\sigma_{c,d}\bar\tau_{\alpha,\mu}=\bar\sigma(1,\alpha,\mu,c,d),\quad \bar\sigma_{c,d}\bar\sigma_\epsilon=\bar\sigma(\epsilon,1,1,\epsilon c,d),$$
$$\bar\tau_{\alpha,\mu}^{-1}\bar\sigma_\epsilon\bar\tau_{\alpha,\mu}=\bar\sigma(\epsilon,\alpha^{1-\epsilon},1,0,0),\quad \bar\sigma_\epsilon\bar\tau_{\alpha,\mu}\bar\sigma_\epsilon=\bar\tau_{\alpha^\epsilon,\mu},$$
$$\bar\sigma_{c,d}^{-1}\bar\tau_{\alpha,\mu}\bar\sigma_{c,d}=\bar\sigma(1,\alpha,\mu,(\mu-1)c,(\mu-1)d),\quad \bar\tau_{\alpha,\mu}^{-1}\bar\sigma_{c,d}\bar\tau_{\alpha,\mu}=\bar\sigma_{\mu^{-1}c,\mu^{-1}d},$$
$$\bar\sigma_{c,d}^{-1}\bar\sigma_\epsilon\bar\sigma_{c,d}=\bar\sigma(\epsilon,1,1,(1-\epsilon)c,0),\quad \bar\sigma_\epsilon\bar\sigma_{c,d}\bar\sigma_\epsilon=\bar\sigma_{\epsilon c,d}.$$

Therefore, $\mathfrak{I}\mathfrak{c}_4$ is an abelian normal subgroup of $\mathrm{Aut}(\mathbf{W}^g(0,1))$ and

$$\mathrm{Aut}(\mathbf{W}^g(0,1))=(\mathfrak{I}\mathfrak{c}_4)\rtimes(\mathfrak{a}_4\ltimes\mathfrak{b}_4)\cong(\mathbb{C}^\infty\rtimes\mathbb{C}^\infty)\rtimes(\mathbb{Z}_2\ltimes(\mathbb{C}^*\times\mathbb{C}^*)).$$

Case 3 $b\neq 0,1$. By [21], we assume

$$\bar\sigma(L_m)=\epsilon\alpha^mL_{\epsilon m}+\alpha^m m\lambda_\epsilon I_{\epsilon m},\quad \bar\sigma(I_m)=\alpha^m\mu I_{\epsilon m},$$
$$\bar\sigma(Y_{m+\frac12})=f_{\epsilon(m+\frac12)-\frac12}Y_{\epsilon(m+\frac12)},\ \forall\, m\in\mathbb{Z}.$$

Similarly, we still have

$$f_{\epsilon(m+\frac12)-\frac12}=\alpha^m\sqrt{\epsilon\alpha\mu},\quad \forall m\in\mathbb{Z}.$$

Therefore

$$\bar\sigma(L_m)=\epsilon\alpha^mL_{\epsilon m}+\alpha^m m(cm+d)I_{\epsilon m},\quad \bar\sigma(I_m)=\alpha^m\mu I_{\epsilon m},\tag{3.11}$$
$$\bar\sigma(Y_{m+\frac12})=\alpha^m\sqrt{\epsilon\alpha\mu}\ Y_{\epsilon(m+\frac12)}\qquad \forall\, m\in\mathbb{Z}.\tag{3.12}$$

By the automorphism group of $\mathcal{I}_{\mathfrak{v}}(0,-1)$ in [7], we have

$$\mathrm{Aut}(\mathbf{W}^g(0,b))\cong(\mathbb{C}^\infty\rtimes\mathbb{C}^\infty)\rtimes(\mathbb{Z}_2\ltimes(\mathbb{C}^*\times\mathbb{C}^*)),$$

where $b\neq 0,1$. □

Lemma 3.4. *For $a=1$, $\mathrm{Aut}(\mathbf{W}^{\mathbf{g}}(a,b))\cong(\mathbb{C}^\infty\rtimes\mathbb{C}^\infty)\rtimes(\mathbb{Z}_2\ltimes(\mathbb{C}^*\times\mathbb{C}^*))$. where $\mathbb{C}^\infty=\{(a_i)_{i\in\mathbb{Z}}\mid a_i\in\mathbb{C},\ \text{all but a finite number of the } a_i \text{ are zero}\}$, $\mathbb{Z}_2=\mathbb{Z}/2\mathbb{Z}$, $\mathbb{C}^*=\mathbb{C}\backslash\{0\}$.*

Proof. For any $\sigma\in\mathrm{Aut}(\mathbf{W}^g(1,b))$, similar to the ideas in lemma 3.2-3.3, there exist $\tau\in\mathfrak{I}$ and $\bar\sigma=\tau^{-1}\sigma$. We may assume that

$$\bar\sigma(L_0)=\epsilon L_0+\lambda_{-1}I_{-1}+l_{-1}Y_{-1+\frac12},\ \bar\sigma(L_n)=\epsilon\alpha^nL_{\epsilon n}+\alpha^n\sum\lambda_{n_i}I_{n_i}+\alpha^n\sum\mu_{m_j}Y_{m_j+\frac12},\ \forall n\neq0,$$
$$\bar\sigma(I_n)=\sum c_pI_p+\sum d_qY_{q+\frac12},\ \bar\sigma(Y_{n+\frac12})=\alpha^n\sum e_uI_u+\sum f_vY_{v+\frac12},\ \forall\, n\in\mathbb{Z},$$

where each $\sum$ formula is of finite terms and $\lambda_{n_i},\mu_{m_j},c_p,d_q,e_u,f_v\in\mathbb{C}$, $n_i,m_j,p,q,u,v\in\mathbb{Z}$, $\alpha\in\mathbb{C}^*,\epsilon=\pm1$. By identities

$$[\bar\sigma(L_0),\bar\sigma(L_n)]=-n\bar\sigma(L_n),\quad [\bar\sigma(L_0),\bar\sigma(I_n)]=-(n+1)\bar\sigma(I_n),$$
$$[\bar\sigma(L_0),\bar\sigma(Y_{n+\frac12})]=-(n+1)\bar\sigma(Y_{n+\frac12}),$$

we have

$$\mu_{m_j}[\epsilon(m_j+1)-n]=0, \quad \lambda_{-1}bnI_{\epsilon n-1}=\sum[\epsilon(n_i+1)-n]\lambda_{n_i}I_{n_i}.$$

$$d_q[\epsilon(q+1)-(n+1)]=0, \quad \epsilon\sum c_p(p+1)I_p+\sum l_{-1}d_q(q+1)I_q=(n+1)\sum c_pI_p,$$

$$e_u[\epsilon(u+1)-(n+1)]=0, \qquad f_v[\epsilon(v+1)-(n+1)]=0.$$

So

$$m_j=\epsilon n-1,\ \lambda_{-1}b=0,\ n_i=\epsilon n-1,$$

$$l_{-1}=0, \quad p=q=\epsilon(n+1)-1, \quad u=v=\epsilon(n+1)-1,$$

and

$$\bar{\sigma}(L_0)=\epsilon L_0+\lambda_{-1}I_{-1}, \quad \bar{\sigma}(L_n)=\epsilon\alpha^nL_{\epsilon n}+\alpha^n\lambda_{\epsilon n-1}I_{\epsilon n-1}+\alpha^n\mu_{\epsilon n-1}Y_{\epsilon n-1+\frac{1}{2}},$$

$$\bar{\sigma}(I_n)=c_{\epsilon(n+1)-1}I_{\epsilon(n+1)-1}+d_{\epsilon(n+1)-1}Y_{\epsilon(n+1)-1+\frac{1}{2}},$$

$$\bar{\sigma}(Y_{n+\frac{1}{2}})=\alpha^ne_{\epsilon(n+1)-1}I_{\epsilon(n+1)-1}+f_{\epsilon(n+1)-1}Y_{\epsilon(n+1)-1+\frac{1}{2}}.$$

On the other hand, for $m\neq n$, by $[\bar{\sigma}(I_m),\bar{\sigma}(I_n)]=0$, we have

$$d_{\epsilon(m+1)-1}d_{\epsilon(n+1)-1}\epsilon(m-n)=0.$$

So

$$d_{\epsilon(n+1)-1}=0$$

for all $n\in\mathbb{Z}$. Consequently, by $[\bar{\sigma}(L_m),\bar{\sigma}(I_n)]=-(n+1+bm)\bar{\sigma}(I_{m+n})$, we have

$$(n+1+bm)(\alpha^mc_{\epsilon(n+1)-1}-c_{\epsilon(m+n+1)-1})=0.$$

We can deduce that

$$c_{\epsilon(m+1)-1}=\alpha^m\mu,$$

So

$$\bar{\sigma}(I_m)=\alpha^m\mu I_{\epsilon(m+1)-1}, \quad \forall m\in\mathbb{Z} \tag{3.13}$$

where μ is a fixed nonzero complex number. For $m\neq n$, by

$$[\bar{\sigma}(L_m),\bar{\sigma}(L_n)]=(m-n)\bar{\sigma}(L_{m+n}), \quad [\bar{\sigma}(L_n),\bar{\sigma}(Y_{m+\frac{1}{2}})]=-(m+1+\frac{bn-n}{2})\bar{\sigma}(Y_{m+n+\frac{1}{2}}),$$

$$[\bar{\sigma}(Y_{m+\frac{1}{2}}),\bar{\sigma}(Y_{n+\frac{1}{2}})]=(m-n)\bar{\sigma}(I_{m+n+1}),$$

we have

$$\mu_{\epsilon m-1}[m+\frac{(b-1)n}{2}]-\mu_{\epsilon n-1}[n+\frac{(b-1)m}{2}]=(m-n)\mu_{\epsilon(m+n)-1}, \tag{3.14}$$

$$\mu_{\epsilon m-1}\mu_{\epsilon n-1}\epsilon(m-n)+\lambda_{\epsilon m-1}(m+bn)-\lambda_{\epsilon n-1}(n+bm)=(m-n)\lambda_{\epsilon(m+n)-1}, \tag{3.15}$$

$$(m+1+\frac{(b-1)n}{2})\alpha^{m+n}e_{\epsilon(m+n+1)-1}-(m+1+bn)\alpha^{m+n}e_{\epsilon(m+1)-1}=\epsilon(m+1-n)\alpha^n\mu_{\epsilon n-1}f_{\epsilon(m+1)-1}, \tag{3.16}$$

$$\alpha^nf_{\epsilon(m+1)-1}(m+1+\frac{(b-1)n}{2})=(m+1+\frac{(b-1)n}{2})f_{\epsilon(m+n+1)-1}, \tag{3.17}$$

$$f_{\epsilon(m+1)-1}f_{\epsilon(n+1)-1}\epsilon = \alpha^{m+n+1}\mu. \tag{3.18}$$

Let $n = 0$ in (3.16) and (3.18), we have

$$\mu_{-1} = 0, \quad f_{\epsilon(m+1)-1} = \epsilon\alpha^{m+1}\mu f_{\epsilon-1}^{-1}, \quad \forall m \in \mathbb{Z},$$

where $f_{\epsilon-1}^{-1} = f$ is a fixed nonzero complex number. From (3.17), we can deduce $f_{\epsilon(n+1)-1} = \alpha^n f_{\epsilon-1}$ for all $n \in \mathbb{Z}$. Taking it into (3.18), we get $f_{\epsilon-1} = \sqrt{\epsilon\alpha\mu}$, then

$$f_{\epsilon(n+1)-1} = \alpha^n\sqrt{\epsilon\alpha\mu}, \quad \forall n \in \mathbb{Z}. \tag{3.19}$$

Case 1: $b = 0$. Let $n = -m, n = 1$ in (3.14) respectively, we get

$$\mu_{\epsilon m-1} + \mu_{\epsilon(-m)-1} = 0, \tag{3.20}$$

$$(2m-1)\mu_{\epsilon m-1} - (2-m)\mu_{\epsilon-1} = (2m-2)\mu_{\epsilon(m+1)-1}, \quad \forall m \in \mathbb{Z}. \tag{3.21}$$

Using induction on $m \geq 2$ in (3.21), we obtain

$$\mu_{\epsilon m-1} = \frac{(2m-3)!!}{2^{m-2}(m-2)!}(\mu_{\epsilon 2-1} - 2\mu_{\epsilon-1}) + m\mu_{\epsilon-1}, \quad m \geq 2. \tag{3.22}$$

On the other hand, let $m = -3, n = 1$ in (3.14) and use (3.20), (3.22), we have $\mu_{\epsilon 2-1} - 2\mu_{\epsilon-1} = 0$. So

$$\mu_{\epsilon m-1} = m\mu_{\epsilon-1}, \quad \forall m \in \mathbb{Z}. \tag{3.23}$$

Let $m = -1$ in (3.16) and use (3.23), (3.19), we obtain

$$e_{\epsilon(n+1)-1} = 2\epsilon(n+1)\mu_{\epsilon-1}\sqrt{\epsilon\alpha\mu}, \quad \forall n \in \mathbb{Z}. \tag{3.24}$$

From (3.15) and (3.23), we have

$$mn\epsilon(m-n)\mu_{\epsilon-1}^2 + m\lambda_{\epsilon m-1} - n\lambda_{\epsilon n-1} = (m-n)\lambda_{\epsilon(m+n)-1}. \tag{3.25}$$

Let $m + n = 0$ in (3.25), we have

$$\lambda_{\epsilon m-1} + \lambda_{\epsilon(-m)-1} = 2\lambda_{-1} + 2m^2\epsilon\mu_{\epsilon-1}^2, \quad \forall m \in \mathbb{Z}. \tag{3.26}$$

Let $n = 1$ in (3.25) and use induction on m, we obtain

$$\lambda_{\epsilon m-1} = (m-1)(\lambda_{\epsilon 2-1} - \lambda_{\epsilon-1}) + (m-1)(m-2)\epsilon\mu_{\epsilon-1}^2 + \lambda_{\epsilon-1}, \quad m \geq 1. \tag{3.27}$$

On the other hand, let $m = -2, n = 1$ and $m = -5, n = 3$ in (3.25) respectively and use (3.26), (3.27), then

$$-4\epsilon\mu_{\epsilon-1}^2 + 2\lambda_{-1} + 2\lambda_{\epsilon 2-1} - 4\lambda_{\epsilon-1} = 0, \quad -4\epsilon\mu_{\epsilon-1}^2 + 3\lambda_{-1} + 3\lambda_{\epsilon 2-1} - 6\lambda_{\epsilon-1} = 0.$$

Combine the two identities, we get

$$\lambda_{-1} = -\lambda_{\epsilon 2-1} + 2\lambda_{\epsilon-1} \quad \text{and} \quad \mu_{\epsilon-1} = 0.$$

So

$$\lambda_{\epsilon m-1} = m(\lambda_{\epsilon-1} - \lambda_{-1}) + \lambda_{-1}, \quad \forall m \in \mathbb{Z}.$$

Set $\lambda_{\epsilon-1}-\lambda_{-1}=c, \lambda_{-1}=d$, then

$$\bar{\sigma}(L_n)=\epsilon\alpha^n L_{\epsilon n}+\alpha^n(nc+d)I_{\epsilon n-1}, \tag{3.28}$$

$$\bar{\sigma}(I_n)=\alpha^n\mu I_{\epsilon(n+1)-1}, \quad \bar{\sigma}(Y_{n+\frac{1}{2}})=\alpha^n\sqrt{\epsilon\alpha\mu}\; Y_{\epsilon(n+1)-1+\frac{1}{2}}, \tag{3.29}$$

for all $n\in\mathbb{Z}$. Where $\alpha,\mu\in\mathbb{C}^*, c,d\in\mathbb{C}, \epsilon=\pm1$. Denote by $\bar{\sigma}(\epsilon,\alpha,\mu,c,d)$ the automorphism of $\mathbf{W}^g(1,0)$ satisfying (3.28) and (3.29), then

$$\bar{\sigma}(\epsilon_1,\alpha_1,\mu_1,c_1,d_1)\bar{\sigma}(\epsilon_2,\alpha_2,\mu_2,c_2,d_2)=\bar{\sigma}(\epsilon_1\epsilon_2,\alpha_1^{\epsilon_2}\alpha_2,\alpha_1^{\epsilon_2-1}\mu_1\mu_2,c_1+\alpha_1^{-1}\mu_1c_2,\epsilon_2d_1+\alpha_1^{-1}\mu_1d_2), \tag{3.30}$$

and $\bar{\sigma}(\epsilon_1,\alpha_1,\mu_1,c_1,d_1)=\bar{\sigma}(\epsilon_2,\alpha_2,\mu_2,c_2,d_2)$ if and only if $\epsilon_1=\epsilon_2$, $\alpha_1=\alpha_2$, $\mu_1=\mu_2$, $c_1=c_2$, $d_1=d_2$. Let

$$\bar{\sigma}_\epsilon=\bar{\sigma}(\epsilon,1,1,0,0), \quad \bar{\tau}_{\alpha,\mu}=\bar{\sigma}(1,\alpha,\mu,0,0), \quad \bar{\sigma}_{c,d}=\bar{\sigma}(1,1,1,c,d)$$

$$\mathfrak{a}_3=\{\bar{\sigma}_\epsilon\mid\epsilon\in\{\pm1\}\}, \quad \mathfrak{b}_3=\{\bar{\tau}_{\alpha,\mu}\mid\alpha,\mu\in\mathbb{C}^*\}, \quad \mathfrak{c}_3=\{\bar{\sigma}_{c,d}\mid c,d\in\mathbb{C}\}.$$

Similar to the discussion in the lemma 3.3, we have the same results

$$\mathrm{Aut}(\mathbf{W}^g(1,0))=(\mathfrak{I}\mathfrak{c}_3)\rtimes(\mathfrak{a}_3\ltimes\mathfrak{b}_3)\cong(\mathbb{C}^\infty\rtimes\mathbb{C}^\infty)\rtimes(\mathbb{Z}_2\ltimes(\mathbb{C}^*\times\mathbb{C}^*)).$$

Case 2: $b=1$. We have $\lambda_{-1}=0$, so $\bar{\sigma}(L_0)=\epsilon L_0$. Let $m+n=0$ in (3.14) and (3.15), we have

$$\mu_{\epsilon m-1}+\mu_{\epsilon(-m)-1} = 0, \tag{3.31}$$

$$\mu_{\epsilon m-1}\mu_{\epsilon(-m)-1} = 0, \tag{3.32}$$

for all $m\in\mathbb{Z}$. From (3.31) and (3.32), we can deduce that

$$\mu_{\epsilon m-1}=0, \quad \forall m\in\mathbb{Z}.$$

Consequently, by (3.15), we have

$$(m+n)(\lambda_{\epsilon m-1}-\lambda_{\epsilon n-1})=(m-n)\lambda_{\epsilon(m+n)-1} \tag{3.33}$$

Let $n=-1$ in (3.33) and use induction on m, we get

$$\lambda_{\epsilon m-1}=\frac{m(m-1)}{2}(\lambda_{\epsilon-1}+\lambda_{-\epsilon-1})+m\lambda_{\epsilon-1}, \qquad \forall m\in\mathbb{Z}.$$

By (3.16), we have

$$(m+1)e_{\epsilon(m+n+1)-1}=(m+n+1)e_{\epsilon(m+1)-1}. \tag{3.34}$$

Let $m=0$ in (3.34), we get

$$e_{\epsilon(n+1)-1}=(n+1)e_{\epsilon-1}, \qquad \forall n\in\mathbb{Z}.$$

Set $c=\dfrac{\lambda_{\epsilon-1}+\lambda_{-\epsilon-1}}{2}$, $d=\lambda_{\epsilon-1}-c$, $e=e_{\epsilon-1}$, then

$$\bar{\sigma}(L_n)=\epsilon\alpha^n L_{\epsilon n}+\alpha^n n(nc+d)I_{\epsilon n-1}, \quad \bar{\sigma}(I_n)=\alpha^n\mu I_{\epsilon(n+1)-1}, \tag{3.35}$$

$$\bar{\sigma}(Y_{n+\frac{1}{2}}) = \alpha^n(n+1)eI_{\epsilon(n+1)-1} + \alpha^n\sqrt{\epsilon\alpha\mu}\ Y_{\epsilon(n+1)-1+\frac{1}{2}}, \tag{3.36}$$

for all $n \in \mathbb{Z}$. Where $c,\ d,\ e \in \mathbb{C},\ \alpha,\ \mu \in \mathbb{C}^*,\ \epsilon = \pm 1$. If $\bar{\sigma}$ is a linear operator on $\mathbf{W}^g(a,b)$ satisfying (3.35) and (3.36) for some $\alpha, \mu \in \mathbb{C}^*$, then it is easy to check that $\bar{\sigma} \in \mathrm{Aut}(W^g(a,b))$. Denote by $\bar{\sigma}(\epsilon,\alpha,\mu,c,d,e)$ the automorphism of $\mathbf{W}^g(1,1)$ satisfying (3.35) and (3.36), then

$$\begin{aligned}&\bar{\sigma}(\epsilon_1,\alpha_1,\mu_1,c_1,d_1,e_1)\bar{\sigma}(\epsilon_2,\alpha_2,\mu_2,c_2,d_2,e_2) \\ &= \bar{\sigma}(\epsilon_1\epsilon_2, \alpha_1^{\epsilon_2}\alpha_2, \alpha_1^{\epsilon_2-1}\mu_1\mu_2, \epsilon_2 c_1 + \alpha_1^{-1}\mu_1 c_2, d_1 + \alpha_1^{-1}\mu_1 d_2, \alpha_1^{\epsilon_2-1}\mu_1 e_2 + \alpha_1^{\epsilon_2-1}\epsilon_2 e_1\sqrt{\epsilon_2\alpha_2\mu_2}),\end{aligned} \tag{3.37}$$

and $\bar{\sigma}(\epsilon_1,\alpha_1,\mu_1,c_1,d_1,e_1) = \bar{\sigma}(\epsilon_2,\alpha_2,\mu_2,c_2,d_2,e_2)$ if and only if $\epsilon_1 = \epsilon_2$, $\alpha_1 = \alpha_2$, $\mu_1 = \mu_2$, $c_1 = c_2$, $d_1 = d_2$, $e_1 = e_2$. Let

$$\bar{\sigma}_\epsilon = \bar{\sigma}(\epsilon,1,1,0,0,0),\quad \bar{\tau}_{\alpha,\mu} = \bar{\sigma}(1,\alpha,\mu,0,0,0),\quad \bar{\sigma}_{c,d,e} = \bar{\sigma}(1,1,1,c,d,e)$$

$$\mathfrak{a}_3 = \{\bar{\sigma}_\epsilon \mid \epsilon \in \{\pm 1\}\},\quad \mathfrak{b}_3 = \{\bar{\tau}_{\alpha,\mu} \mid \alpha,\mu \in \mathbb{C}^*\},\quad \mathfrak{c}_3 = \{\bar{\sigma}_{c,d,e} \mid c,d,e \in \mathbb{C}\}.$$

From (3.37), we can see that $\mathfrak{a}_3$, $\mathfrak{b}_3$, $\mathfrak{c}_3$, are subgroups of $\mathrm{Aut}(\mathbf{W}^g(1,1))$. Similar to the discussion in lemma 3.3 and [22], we have

$$\mathrm{Aut}(\mathbf{W}^g(1,1)) = (\mathfrak{I}\mathfrak{c}_3) \rtimes (\mathfrak{a}_3 \ltimes \mathfrak{b}_3) \cong (\mathbb{C}^\infty \rtimes \mathbb{C}^\infty) \rtimes (\mathbb{Z}_2 \ltimes (\mathbb{C}^* \times \mathbb{C}^*)).$$

Case 3: $b \neq 0, 1$. We always have $\lambda_{-1} = 0$. Let $n = -m, n = 1$ in (3.14) respectively, we have

$$\begin{aligned}(3-b)(\mu_{\epsilon m-1} + \mu_{\epsilon(-m)-1}) &= 0, \\ (2m+b-1)\mu_{\epsilon m-1} - [2+(b-1)m]\mu_{\epsilon-1} &= 2(m-1)\mu_{\epsilon(m+1)-1}.\end{aligned} \tag{3.38}$$

That is

$$2(m-1)[\mu_{\epsilon(m+1)-1} - (m+1)\mu_{\epsilon-1}] = (2m+b-1)[\mu_{\epsilon m-1} - m\mu_{\epsilon-1}]. \tag{3.39}$$

Subcase 3.1: $b \neq 3$. From (3.38), we have

$$\mu_{\epsilon(-m)-1} = -\mu_{\epsilon m-1},\quad \forall m \in \mathbb{Z}. \tag{3.40}$$

So using induction on $m \geq 2$ in (3.39), we obtain

$$\mu_{\epsilon m-1} = \frac{(3+b)(5+b)\cdots(2m-3+b)}{(2m-4)!!}(\mu_{\epsilon 2-1} - 2\mu_{\epsilon-1}) + m\mu_{\epsilon-1}, m \geq 3. \tag{3.41}$$

On the other hand, let $m = 3, n = -1$ in (3.14) and use (3.40), (3.41) we have

$$(5+4b-b^2)\mu_{\epsilon 2-1} = 2(5+4b-b^2)\mu_{\epsilon-1}. \tag{3.42}$$

If $5+4b-b^2 \neq 0$, that is $b \neq -1$ and $b \neq 5$, then $\mu_{\epsilon 2-1} = 2\mu_{\epsilon-1}$. Consequently,

$$\mu_{\epsilon m-1} = m\mu_{\epsilon-1},\quad \forall m \in \mathbb{Z}.$$

From (3.15), we have

$$mn\epsilon(m-n)\mu_{\epsilon-1}^2 + \lambda_{\epsilon m-1}(m+bn) - \lambda_{\epsilon n-1}(n+bm) = (m-n)\lambda_{\epsilon(n+m)-1}. \tag{3.43}$$

Let $n=-1, m+n=0$ in (3.43) respectively, we get

$$-m(m+1)\epsilon\mu_{\epsilon-1}^2+\lambda_{\epsilon m-1}(m-b)-\lambda_{\epsilon(-1)-1}(-1+bm)=(m+1)\lambda_{\epsilon(m-1)-1} \quad (3.44)$$

$$(1-b)(\lambda_{\epsilon m-1}+\lambda_{\epsilon(-m)-1})=2\epsilon m^2\mu_{\epsilon-1}^2 \quad (3.45)$$

Let $m=2$ in (3.44) and use (3.45), we have

$$(-4+2b)\epsilon\mu_{\epsilon-1}^2+(2-b)(1-b)\lambda_{\epsilon 2-1}=(1-b)(4-2b)\lambda_{\epsilon-1} \quad (3.46)$$

If $b\neq 2$, we have

$$2\epsilon\mu_{\epsilon-1}^2+2(1-b)\lambda_{\epsilon-1}=(1-b)\lambda_{\epsilon 2-1} \quad (3.47)$$

Using (3.45), we obtain

$$\lambda_{\epsilon 2-1}=3\lambda_{\epsilon-1}+\lambda_{\epsilon(-1)-1}. \quad (3.48)$$

Similarly, we also get

$$\lambda_{\epsilon 3-1}=6\lambda_{\epsilon-1}+3\lambda_{\epsilon(-1)-1}$$

Then induction on m in (3.44) and using (3.45) and (3.48), we have

$$\lambda_{\epsilon m-1}=\frac{m(m+1)}{2}\lambda_{\epsilon-1}+\frac{m(m-1)}{2}\lambda_{\epsilon(-1)-1}=\frac{m(m-1)}{2}[\lambda_{\epsilon-1}+\lambda_{\epsilon(-1)-1}]+m\lambda_{\epsilon-1},\ \forall m\in\mathbb{Z}.$$

Let $m=0, n=1$ and $m=-1$ in (3.16) respectively, we have

$$(b+1)(e_{\epsilon 2-1}-2e_{\epsilon-1})=0 \quad (3.49)$$

and

$$\frac{b-1}{2}e_{\epsilon n-1}-be_{-1}=-\epsilon\alpha\mu_{\epsilon n-1}f_{-1},\quad n\neq 0. \quad (3.50)$$

From the two identities, we have $e_{-1}=0$ if $b\neq -1$. So

$$e_{\epsilon n-1}=\frac{-2\epsilon n\alpha\mu_{\epsilon-1}f_{-1}}{b-1},\quad \forall n\in\mathbb{Z}.$$

Consequently, by (3.19), we get

$$e_{\epsilon(n+1)-1}=\frac{-2\epsilon(n+1)\mu_{\epsilon-1}\sqrt{\epsilon\alpha\mu}}{b-1},\quad \forall n\in\mathbb{Z}.$$

Set $\frac{\lambda_{\epsilon-1}+\lambda_{\epsilon(-1)-1}}{2}=c, \lambda_{\epsilon-1}=d$, then by (3.45), $\mu_{\epsilon-1}=\sqrt{\epsilon(1-b)c}$. So we get

$$\bar{\sigma}(L_n)=\epsilon\alpha^n L_{\epsilon n}+\alpha^n n((n-1)c+d)I_{\epsilon n-1}+\alpha^n n\sqrt{\epsilon(1-b)c}Y_{\epsilon n-1+\frac{1}{2}}, \quad (3.51)$$

$$\bar{\sigma}(I_n)=\alpha^n\mu I_{\epsilon(n+1)-1}, \quad (3.52)$$

$$\bar{\sigma}(Y_{n+\frac{1}{2}})=\alpha^n\frac{-2\epsilon(n+1)\sqrt{(1-b)c\alpha\mu}}{b-1}I_{\epsilon(n+1)-1}+\alpha^n\sqrt{\epsilon\alpha\mu}\ Y_{\epsilon(n+1)-1+\frac{1}{2}}, \quad (3.53)$$

for all $n\in\mathbb{Z}$. Where $c,\ d\in\mathbb{C},\ \alpha,\ \mu\in\mathbb{C}^*,\ \epsilon=\pm 1$. Similar to the discussion in lemma 3.3, we have

$$\mathrm{Aut}(\mathbf{W}^g(1,b))\cong(\mathbb{C}^\infty\rtimes\mathbb{C}^\infty)\rtimes(\mathbb{Z}_2\ltimes(\mathbb{C}^*\times\mathbb{C}^*)),\quad b\neq 0,1,2,3,-1,5.$$

If $b=2$, Induction on m in (3.44) and using (3.45), we have

$$\lambda_{\epsilon m-1}=\frac{m(m-1)(m+1)}{6}\lambda_{\epsilon 2-1}-\frac{m(m-1)(m-2)}{6}[\lambda_{\epsilon-1}+\lambda_{\epsilon(-1)-1}]-\frac{m(m-2)(m+2)}{3}\lambda_{\epsilon-1},$$

for all $m\in\mathbb{Z}$. Similar to the above discussion, we have

$$\mathrm{Aut}(\mathbf{W}^g(1,2))\cong(\mathbb{C}^\infty\rtimes\mathbb{C}^\infty)\rtimes(\mathbb{Z}_2\ltimes(\mathbb{C}^*\times\mathbb{C}^*)).$$

If $b=-1$, by (3.14), we have

$$\mu_{\epsilon m-1}+\mu_{\epsilon n-1}=\mu_{\epsilon(m+n)-1}. \tag{3.54}$$

Let $n=1$ in (3.54) and induction on $m>1$, according to (3.40), we still get $\mu_{\epsilon m-1}=m\mu_{\epsilon-1}$ for all $m\in\mathbb{Z}$. By (3.50), we obtain $e_{\epsilon n-1}=\epsilon\alpha n\mu_{\epsilon-1}f_{-1}+e_{-1}$ for all $n\in\mathbb{Z}$. We still have the following result

$$\mathrm{Aut}(\mathbf{W}^g(1,-1))\cong(\mathbb{C}^\infty\rtimes\mathbb{C}^\infty)\rtimes(\mathbb{Z}_2\ltimes(\mathbb{C}^*\times\mathbb{C}^*)).$$

If $b=5$, by (3.14), we get

$$\mu_{\epsilon m-1}=\frac{m(m-1)(m+1)}{6}(\mu_{\epsilon 2-1}-2\mu_{\epsilon-1})+m\mu_{\epsilon-1},\quad\forall m\in\mathbb{Z}.$$

Let $m+n=0, n=1$ in (3.15) respectively, for all $m\in\mathbb{Z}$, we have

$$2(\lambda_{\epsilon m-1}+\lambda_{\epsilon(-m)-1})=-\epsilon\mu_{\epsilon m-1}^2, \tag{3.55}$$

$$\epsilon(m-1)\mu_{\epsilon m-1}\mu_{\epsilon-1}+(m+5)\lambda_{\epsilon m-1}-(1+5m)\lambda_{\epsilon-1}=(m-1)\lambda_{\epsilon(m+1)-1}. \tag{3.56}$$

Let $m=\pm 2$ in (3.56) respectively and use (3.55), we obtain

$$\lambda_{\epsilon 2-1}=-\frac{\epsilon(\mu_{\epsilon 2-1}-\mu_{\epsilon-1})^2}{2}+2\lambda_{\epsilon-1}, \tag{3.57}$$

$$\lambda_{\epsilon 3-1}=\epsilon\mu_{\epsilon 2-1}\mu_{\epsilon-1}-\frac{7}{2}\epsilon(\mu_{\epsilon 2-1}-\mu_{\epsilon-1})^2+3\lambda_{\epsilon-1}. \tag{3.58}$$

On the other hand, let $m=-3$ in (3.56) and use (3.55), (3.57), we have

$$\lambda_{\epsilon 3-1}=26\epsilon\mu_{\epsilon 2-1}\mu_{\epsilon-1}-\frac{43}{2}\epsilon\mu_{\epsilon-1}^2-8\epsilon\mu_{\epsilon 2-1}^2+3\lambda_{\epsilon-1}. \tag{3.59}$$

Combine (3.58) and (3.59), we still get

$$\mu_{\epsilon 2-1}=2\mu_{\epsilon-1}.$$

So

$$\mu_{\epsilon m-1}=m\mu_{\epsilon-1},\quad\forall m\in\mathbb{Z}.$$

Consequently, we get the complete same results as the above lemma, that is

$$\mathrm{Aut}(\mathbf{W}^g(1,5))\cong(\mathbb{C}^\infty\rtimes\mathbb{C}^\infty)\rtimes(\mathbb{Z}_2\ltimes(\mathbb{C}^*\times\mathbb{C}^*)).$$

Subcase 3.2: $b = 3$. Letting $n = \pm 1$ in (3.14) respectively and using induction on m, we can deduce

$$\mu_{\epsilon m-1} = \frac{m(m-1)}{2}(\mu_{\epsilon-1} + \mu_{\epsilon(-1)-1}) + m\mu_{\epsilon-1}, \ \forall m \in \mathbb{Z}. \tag{3.60}$$

Let $m + n = 0, n = 1$ in (3.15) respectively, for all $m \in \mathbb{Z}$, we have

$$\lambda_{\epsilon m-1} + \lambda_{\epsilon(-m)-1} = \epsilon\mu_{\epsilon m-1}\mu_{\epsilon(-m)-1}, \tag{3.61}$$

$$\epsilon(m-1)\mu_{\epsilon m-1}\mu_{\epsilon-1} + (m+3)\lambda_{\epsilon m-1} - (1+3m)\lambda_{\epsilon-1} = (m-1)\lambda_{\epsilon(m+1)-1}. \tag{3.62}$$

Induction on $m > 2$ in (3.62) and computation by computer, we obtain

$$\lambda_{\epsilon m-1} = m\lambda_{\epsilon-1} - \frac{m(m-1)}{2}\epsilon\mu^2_{\epsilon-1} + \frac{m(m^3+m-2)}{4}\epsilon(\mu_{\epsilon-1} + \mu_{\epsilon(-1)-1})\mu_{\epsilon-1}, \ m > 2. \tag{3.63}$$

At the same time, let $m = -3n$ in (3.15), we have

$$\lambda_{\epsilon(-2n)-1} = -2\lambda_{\epsilon n-1} + \epsilon\mu_{\epsilon(-3n)-1}\mu_{\epsilon n-1}, \quad \forall n \in \mathbb{Z}. \tag{3.64}$$

Let $n = -2$ and $n = 1$ in (3.64) and use (3.60), we have

$$\lambda_{\epsilon 4-1} = 4\lambda_{\epsilon-1} + 15\epsilon\mu^2_{\epsilon-1} + 45\epsilon\mu^2_{\epsilon(-1)-1} + 66\epsilon\mu_{\epsilon-1}\mu_{\epsilon(-1)-1}. \tag{3.65}$$

On the other hand, by (3.63), we have

$$\lambda_{\epsilon 4-1} = 4\lambda_{\epsilon-1} + 60\epsilon\mu^2_{\epsilon-1} + 66\epsilon\mu_{\epsilon-1}\mu_{\epsilon(-1)-1}. \tag{3.66}$$

Compare (3.65) with (3.66), we obtain

$$\mu^2_{\epsilon-1} = \mu^2_{\epsilon(-1)-1}$$

If $\mu_{\epsilon-1} = -\mu_{\epsilon(-1)-1}$, by (3.60), (3.61) and (3.63), we have

$$\mu_{\epsilon m-1} = m\mu_{\epsilon-1}, \quad \lambda_{\epsilon m-1} = m\lambda_{\epsilon-1} + \frac{m(m-1)}{2}(\lambda_{\epsilon-1} + \lambda_{\epsilon(-1)-1}), \ \forall m \in \mathbb{Z}.$$

Similar to the discussion in lemma 3.3, we get

$$\mathrm{Aut}(\mathbf{W}^g(1,3)) \cong (\mathbb{C}^\infty \rtimes \mathbb{C}^\infty) \rtimes (\mathbb{Z}_2 \ltimes (\mathbb{C}^* \times \mathbb{C}^*)).$$

If $\mu_{\epsilon-1} = \mu_{\epsilon(-1)-1}$, by (3.60), (3.61) and (3.63), we have

$$\mu_{\epsilon m-1} = m^2\mu_{\epsilon-1}, \quad \lambda_{\epsilon m-1} = m\lambda_{\epsilon-1} + \frac{m(m^3-1)}{2}(\lambda_{\epsilon-1} + \lambda_{\epsilon(-1)-1}), \ \forall m \in \mathbb{Z}.$$

Set $\frac{\lambda_{\epsilon-1} + \lambda_{\epsilon(-1)-1}}{2} = c$, $\frac{\lambda_{\epsilon-1} - \lambda_{\epsilon(-1)-1}}{2} = d$, then $\mu_{\epsilon-1} = \sqrt{2c\epsilon}$. Consequently, we have

$$\bar{\sigma}(L_n) = \epsilon\alpha^n L_{\epsilon n} + \alpha^n n(n^3 c + d)I_{\epsilon n-1} + \alpha^n n^2\sqrt{2\epsilon c}Y_{\epsilon n-1+\frac{1}{2}}, \tag{3.67}$$

$$\bar{\sigma}(I_n) = \alpha^n \mu I_{\epsilon(n+1)-1}, \tag{3.68}$$

$$\bar{\sigma}(Y_{n+\frac{1}{2}}) = -\alpha^n\epsilon(n+1)\sqrt{2c\alpha\mu}I_{\epsilon(n+1)-1} + \alpha^n\sqrt{\epsilon\alpha\mu}\ Y_{\epsilon(n+1)-1+\frac{1}{2}}, \tag{3.69}$$

for all $n \in \mathbb{Z}$, where $c,\ d \in \mathbb{C},\ \alpha,\ \mu \in \mathbb{C}^*,\ \epsilon = \pm 1$. Similar to the discussion in lemma 3.3, we still have

$$\mathrm{Aut}(\mathbf{W}^g(1,3)) \cong (\mathbb{C}^\infty \rtimes \mathbb{C}^\infty) \rtimes (\mathbb{Z}_2 \ltimes (\mathbb{C}^* \times \mathbb{C}^*)).$$

□

By lemma 3.2-3.4, we get the main theorem of this section.

Theorem 3.5.

$$\mathrm{Aut}(\mathbf{W}^{\mathbf{g}}(a,b)) \cong \begin{cases} (\mathbb{C}^\infty \rtimes \mathbb{C}^\infty) \rtimes (\mathbb{C}^* \times \mathbb{C}^*), & if\ a \notin \mathbb{Z}, \\ (\mathbb{C}^\infty \rtimes \mathbb{C}^\infty) \rtimes (\mathbb{Z}_2 \ltimes (\mathbb{C}^* \times \mathbb{C}^*)), & otherwise. \end{cases}$$

where $\mathbb{C}^\infty = \{(a_i)_{i\in\mathbb{Z}} \mid a_i \in \mathbb{C},\ \text{all but a finite number of the } a_i \text{ are zero}\}$, *and* $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$.

□

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